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Nonlinear Self-Duality in Even Dimensions

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Abstract

We show that the Born-Infeld theory with n complex abelian gauge fields written in an auxiliary field formulation has a $U(n,n)$ duality group. We conjecture the form of the Lagrangian obtained by eliminating the auxiliary fields and then introduce a new reality structure leading to a Born-Infeld theory with n real gauge fields and an $Sp(2n,\mathbb{R})$ duality symmetry. The real and complex constructions are extended to arbitrary even dimensions. The maximal noncompact duality group is $U(n,n)$ for complex fields. For real fields the duality group is $Sp(2n,\mathbb{R})$ if half of the dimension of space-time is even and $O(n,n)$ if it is odd. We also discuss duality under the maximal compact subgroup, which is the self-duality group of the theory obtained by fixing the expectation value of a scalar field. Supersymmetric versions of self-dual theories in four dimensions are also discussed.

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1 Introduction

Shortly after the appearance of duality in extended supergravity [1, 2] the theory of duality invariance of theories with abelian gauge fields was developed in [3, 4]. However, there are very few examples of duality invariant interacting gauge theories where the Lagrangian is known in closed form. The most famous is the Born-Infeld theory [5, 6, 7, 8, 9, 10] and in this paper we study its generalization to more than one abelian gauge field.

In Section 2 we present in some detail the theory of duality invariance for a theory of complex gauge fields with holomorphic duality transformations. This is an extension of the theory of duality invariance developed in [3, 4] and was briefly discussed in [11]. However, the duality group can be larger than that presented in [11]. In fact, for a gauge theory with n complex gauge fields the largest possible duality group is $U(n,n)$. We also discuss how to obtain such a theory from a theory with a $U(n) \times U(n)$ duality group, which is the maximal compact subgroup of $U(n,n)$, by introducing an additional n-dimensional matrix valued scalar field.

In Section 3 we describe the Born-Infeld Lagrangian introduced in [11] and written in terms of auxiliary fields. Its form is closely related to the Lagrangian introduced in [12, 13] but differs in two ways. We use a different reality structure for our fields and introduce a dynamical scalar field such that the duality group is extended to a noncompact group.

In Section 4 we discuss the elimination of the auxiliary fields. We have not been able to solve analytically the nonlinear matrix equations obtained from the variation of the auxiliary fields. However we have calculated the first few terms in the perturbative expansion of the Lagrangian in the field strength and based on these we have conjectured in [11] the form of the Lagrangian to all orders. In [11] the conjecture was checked by hand up to the sixth order. It has now been checked by computer up to the seventeenth order. In Appendix A we discuss an equivalent perturbative expansion of the Lagrangian which simplifies the order by order check of the conjecture.

In the theory with auxiliary fields it does not seem possible to work with

real gauge fields, but this can be done in the Lagrangian with the auxiliary fields eliminated. As will be shown in Section 5 this leads to a Born-Infeld theory with an $Sp(2n, \mathbb{R})$ duality group. Assuming that the conjecture of Section 4 is correct, this would be the first example of an interacting gauge theory whose Lagrangian is known to all orders and whose duality group is as large as the duality group of the Maxwell theory with the same number of gauge fields.

In Section 6 we show how to supersymmetrize the Born-Infeld Lagrangian in the formulation with auxiliary fields. We also present the form without auxiliary fields of the supersymmetric Born-Infeld Lagrangian with a single gauge field and a scalar field; this theory is invariant under $SL(2, \mathbb{R})$ duality, which reduces to $U(1)$ duality if the value of the scalar field is suitably fixed. Versions of this theory without the scalar field were presented in [14, 15, 16].

In Section 7 we generalize our construction to arbitrary even dimensions by using antisymmetric tensor fields such that the rank of their field strength equals half the dimension of space-time. We consider first theories with a $U(n, n)$ duality group using complex antisymmetric tensor fields; then we discuss theories with real antisymmetric tensor fields. These have an $Sp(2n, \mathbb{R})$ duality group if half of the space-time dimension is even and $O(n, n)$ if it is odd. The fact that the duality group depends on half the dimension of space-time was discussed earlier in [17, 18, 19, 20, 21].

Finally in Appendix B we briefly discuss two parametrizations of the coset space $U(n, n)/U(n) \times U(n)$ and show how the left multiplication on $U(n, n)$ induces fractional transformations in one of the parametrizations. We also discuss the corresponding coset spaces of the symplectic and orthogonal group and describe their global structure.

2 Duality Invariance

In this section we describe how the theory of self-duality introduced in [3, 4] is modified when we consider complex abelian gauge fields. We only consider a linear action of the duality group which mixes the field strengths and

their duals but not their complex conjugates. We will refer to this as a holomorphic action. Under these conditions the largest allowed duality group is $U(n,n)$ where n is the number of complex gauge fields. If we do not require a holomorphic action, n complex gauge fields are equivalent to $2n$ real gauge fields in which case the largest possible duality group is $Sp(4n, \mathbb{R})$. Later, in Section 5, we will also introduce a Born-Infeld action with real gauge fields which we conjecture to have the largest allowed duality group given the number of gauge fields. However, the argument leading to this conjecture involves Lagrangians with complex gauge fields.

Consider a theory of n complex abelian gauge fields and a scalar field S which is an n-dimensional complex matrix. Here we do not require S to be symmetric and as a result we find a larger duality group than the one appearing in [11]. The gauge fields only enter in the Lagrangian through their field strengths F^a, where $a = 1, \ldots, n$, and their complex conjugates $\bar{F}^a$

$$L = L(F^a, \bar{F}^a, S, \ldots) \ . \tag{1}$$

The dots in (1) represent possible auxiliary fields which could also be present in L. As we will show later, with the scalar field S present the duality group is noncompact while without the scalar field only the maximal compact subgroup survives. We can also add to this Lagrangian a kinetic term for the scalar field S. As explained in [3] additional physical fields, e.g. spinors, can also be introduced, but we shall not consider them explicitly in this paper except in Section 6 where the supersymmetric Born-Infeld theory is discussed.

The dual field strength, or rather the Hodge dual of the dual field strength, $\widetilde{G}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{a\,\rho\sigma}$, is defined as

$$\tilde{G}^a_{\mu\nu} \equiv 2\frac{\partial L}{\partial \bar{F}^{a\,\mu\nu}} \ , \quad \tilde{\bar{G}}^a_{\mu\nu} \equiv 2\frac{\partial L}{\partial F^{a\,\mu\nu}} \ . \tag{2}$$

Throughout this paper we will assume that we are in four space-time dimensions, except in Section 7, where we will show how to generalize our results to theories in even space-time dimensions.

The equations of motion and Bianchi identities transform covariantly under the following holomorphic infinitesimal transformations

$$\delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix} \ . \tag{3}$$

Let ϕ denote all the scalar fields appearing in the Lagrangian and $\phi_\mu = \partial_\mu \phi$. The infinitesimal transformations of the scalar fields are given by

$$\delta \phi^i = \xi^i(\phi) \ , \tag{4}$$

where ξ^i are components of a vector field on the scalar field space. The most general Lagrangian, neglecting possible fermionic fields, has the form $L(F,\, \bar{F},\, \phi,\, \phi_\mu)$. Its variation under (3)(4) can be written as

$$\delta L = \left[\delta_\phi + (GC^T + FD^T) \frac{\partial}{\partial F} + (\bar{G}C^\dagger + \bar{F}D^\dagger) \frac{\partial}{\partial \bar{F}} \right] L \ ,$$

where $\delta_\phi L$ is given by

$$\delta_\phi L = (\xi^i \frac{\partial}{\partial \phi^i} + \phi^j_\mu \frac{\partial \xi^i}{\partial \phi^j} \frac{\partial}{\partial \phi^i_\mu}) L \ .$$

The variation of the Lagrangian must satisfy certain consistency conditions. First note that

$$\frac{\partial}{\partial F} (\delta L) = \delta \left(\frac{\partial L}{\partial F} \right) + \frac{\partial G}{\partial F} C^T \frac{\partial L}{\partial F} + D^T \frac{\partial L}{\partial F} + \frac{\partial \bar{G}}{\partial F} C^\dagger \frac{\partial L}{\partial \bar{F}} \ .$$

Using (2) we obtain

$$\delta \tilde{\bar{G}} = 2 \frac{\partial}{\partial F} (\delta L) - \bar{G}\, C \frac{\partial \tilde{G}}{\partial F} - \frac{\partial \bar{G}}{\partial F} C^\dagger \tilde{G} - \tilde{\bar{G}}\, D \ , \tag{5}$$

and this should be consistent with the variation obtained from (3)

$$\delta \tilde{\bar{G}} = \tilde{\bar{G}}\, A^\dagger + \tilde{\bar{F}}\, B^\dagger \ . \tag{6}$$

Equating (5) and (6) we obtain the consistency condition

$$\frac{\partial}{\partial F} \left(\delta L - \frac{1}{4} \bar{G}(C^\dagger + C)\tilde{G} - \frac{1}{4} \bar{F}(B^\dagger + B)\tilde{F} \right) = \tag{7}$$

$$\frac{\partial L}{\partial F}(D + A^\dagger) + \frac{1}{4} \bar{G}(C - C^\dagger) \frac{\partial \tilde{G}}{\partial F} - \frac{1}{4} \frac{\partial \bar{G}}{\partial F}(C - C^\dagger) G + \frac{1}{4} \tilde{\bar{F}}(B^\dagger - B) \ .$$

The right hand side of the above equation must be a total derivative since the left hand side is one. This is possible if

$$A^\dagger + D = \varepsilon I \ , \quad B^\dagger = B \ , \quad C^\dagger = C \ ,$$

where ε is a real parameter. These are the relations of the fundamental representation of the $U(n,n) \times \mathbb{R}^\star$ Lie algebra[a]. We will only consider the case when ε vanishes. Thus we assume

$$A^\dagger = -D \ , \quad B^\dagger = B \ , \quad C^\dagger = C \ . \tag{8}$$

The relations (8) define the fundamental representation of the Lie algebra of $U(n,n)$. However, in general the transformations (4) of the scalar fields can be implemented only for a subgroup H of $U(n,n)$. The duality group H depends both on the field content and the nature of the interactions of the scalar fields.

Using (8) the consistency condition (7) can be written as

$$\frac{\partial}{\partial F}\left(\delta L - \frac{1}{2}\bar{F}B\widetilde{F} - \frac{1}{2}\bar{G}C\widetilde{G}\right) = 0 \ . \tag{9}$$

Another consistency condition is obtained by applying the Euler operator

$$\frac{\partial}{\partial \phi^i} - \partial_\mu \frac{\partial}{\partial \phi^i_\mu}$$

on the variation of the Lagrangian. Similarly to a derivation in [3], and assuming (8) we obtain

$$\left(\frac{\partial}{\partial \phi^i} - \partial_\mu \frac{\partial}{\partial \phi^i_\mu}\right)\left(\delta L - \frac{1}{2}\bar{F}B\widetilde{F} - \frac{1}{2}\bar{G}C\widetilde{G}\right) = \delta E_i + \frac{\partial \xi^j}{\partial \phi^i} E_j \ , \tag{10}$$

where E_i is the left hand side of the equation of motion for the field ϕ^i

$$E_i = \frac{\partial L}{\partial \phi^i} - \partial_\mu \frac{\partial L}{\partial \phi^i_\mu} \ .$$

[a] $\mathbb{R}^\star$ denotes the group of nonvanishing real numbers.

A sufficient condition to satisfy the consistency equation (9) is given by

$$\delta L = \frac{1}{2}(\bar{F}B\widetilde{F} + \bar{G}C\widetilde{G}) \ . \tag{11}$$

This is equivalent to the invariance of the following combination

$$L - \frac{1}{4}\bar{F}\widetilde{G} - \frac{1}{4}F\widetilde{\bar{G}} \ . \tag{12}$$

Using (11) in (10) we obtain

$$\delta E_i = -\frac{\partial \xi^j}{\partial \phi^i} E_j \ , \tag{13}$$

showing that the equations of motion for the scalar fields form a multiplet under the duality group H. In the examples discussed in this paper the duality group will be $U(n,n)$ for complex gauge fields and $Sp(2n,\mathbb{R})$ for real gauge fields. Ignoring a possible $\mathbb{R}^\star$ factor, present only for a nonvanishing ε, we will refer to these as the maximal noncompact duality groups.

The corresponding finite duality transformations are given by

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = M \begin{pmatrix} G \\ F \end{pmatrix} \ . \tag{14}$$

Here M is an $U(n,n)$ matrix satisfying

$$M^\dagger \, \mathbb{K} \, M = \mathbb{K} \ , \tag{15}$$

where M and $\mathbb{K}$ have the block form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad \mathbb{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Note that the invariant $\mathbb{K}$ defining $U(n,n)$ is the usual off diagonal symplectic form. This explains the similarity of our results with the real case discussed in [3]. One can check that (15) implies the following relations for the block components of M

$$c^\dagger a = a^\dagger c \ , \quad b^\dagger d = d^\dagger b \ , \quad d^\dagger a - b^\dagger c = 1 \ . \tag{16}$$

The infinitesimal relations (8) can be obtained from the finite relations (16) using

$$a \approx 1 + A \ , \quad b \approx B \ , \quad c \approx C \ , \quad d \approx 1 + D \ .$$

In much of this paper we consider Lagrangians which do not depend on the scalar field S, i.e. they depend only on the gauge field strengths and perhaps auxiliary scalar fields, and are invariant only under the maximal compact subgroup $U(n) \times U(n)$ of $U(n,n)$. Then there is a way to introduce the scalar field S which extends the duality group to $U(n,n)$. The maximal compact subgroup $U(n) \times U(n)$ is the subgroup of $U(n,n)$ obtained by requiring (16) and

$$a = d \ , \quad b = -c \ .$$

The corresponding infinitesimal relations are (8) and

$$A = D \ , \quad B = -C \ .$$

Let $\mathcal{L}(F, \bar{F})$ be a Lagrangian describing a theory invariant under $U(n) \times U(n)$, where we suppress the dependence on the auxiliary fields. Then we define a new Lagrangian

$$L(F, \bar{F}, S_1, R, R^\dagger) \equiv \mathcal{L}(RF, \bar{F}R^\dagger) + \frac{1}{2} \operatorname{Tr}(S_1 \tilde{F} \bar{F}) \ , \tag{17}$$

where S_1 is a hermitian n-dimensional matrix and R is a nondegenerate n-dimensional matrix. This Lagrangian describes a theory invariant under $U(n,n)$ if we transform the scalar fields S_1 and R as discussed below. As we will see, the duality invariance of the theory described by $\mathcal{L}$ implies that L depends on R and $R^\dagger$ only through the hermitian positive definite matrix

$$S_2 = R^\dagger R \ . \tag{18}$$

We also define $S \equiv S_1 + iS_2$. Under the duality group S transforms by fractional transformation

$$S' = (aS + b)(cS + d)^{-1} \ , \tag{19}$$

whose infinitesimal form is

$$\delta S = B + AS - SD - SCS \ . \tag{20}$$

It is also convenient to write down the transformation of S_2

$$S_2' = (cS + d)^{-\dagger} S_2 (cS + d)^{-1} \ . \tag{21}$$

In (21) and below we use the notation $-\dagger$ for the hermitian conjugate of the inverse.

Next we show that the Lagrangian L defined in (17) corresponds to a $U(n, n)$ duality invariant theory. We follow closely [21] where the case of real gauge fields was considered. The proof in [21] generalizes the introduction of a single complex scalar field for a $U(1)$ interacting theory discussed in [9, 10]. Using the fact that $\mathcal{L}(F, \bar{F})$ satisfies (11) with compact duality rotations we have

$$\bar{G}^a \widetilde{G}^b + \bar{F}^a \widetilde{F}^b = 0 \ , \tag{22}$$

$$\bar{G}^a \widetilde{F}^b - \bar{F}^a \widetilde{G}^b = 0 \ . \tag{23}$$

The relation (22) corresponds to transformations with $A = 0$ while (23) is obtained by setting $C = 0$. We now introduce some convenient notation

$$\mathcal{F} = RF \ , \quad \widetilde{\bar{\mathcal{G}}} = 2 \frac{\partial \mathcal{L}(\mathcal{F}, \bar{\mathcal{F}})}{\partial \mathcal{F}} \ . \tag{24}$$

Given a Lagrangian $\mathcal{L}$ which depends on F but not its derivatives, we may rewrite (22) and (23) as

$$\bar{\mathcal{G}}^a \widetilde{\mathcal{G}}^b + \bar{\mathcal{F}}^a \widetilde{\mathcal{F}}^b = 0 \ , \tag{25}$$

$$\bar{\mathcal{G}}^a \widetilde{\mathcal{F}}^b - \bar{\mathcal{F}}^a \widetilde{\mathcal{G}}^b = 0 \ . \tag{26}$$

We would like to show that under an infinitesimal $U(n, n)$ duality transformation the change in the Lagrangian L defined in (17) satisfies the duality condition (11)

$$(\delta_F + \delta_{\bar{F}} + \delta_{S_1} + \delta_R + \delta_{R^\dagger}) L = \frac{1}{2} (\bar{F} B \widetilde{F} + \bar{G} C \widetilde{G}) \ . \tag{27}$$

A transformation law for R which is consistent with the relation $R^\dagger R = S_2$ and the duality transformation (21) of S_2 is given by

$$R' = R(cS + d)^{-1},$$

whose infinitesimal transformation is $\delta R = -R(CS + D)$. This choice is somewhat arbitrary since equation (23) is equivalent to the Lagrangian $\mathcal{L}$ being invariant under left multiplication of the gauge field strength by unitary matrices U

$$\mathcal{L}(UF, \bar{F}U^\dagger) = \mathcal{L}(F, \bar{F}) \ .$$

This ensures that left multiplication of R by a unitary matrix leaves the Lagrangian L invariant. It follows that the Lagrangian L only depends on S_2 and not on the specific R chosen[b], as we have already mentioned. Any variation of the form $\delta R = \Omega R - R(CS + D)$, where Ω is anti-hermitian, would still preserve the relation $R^\dagger R = S_2$.

Using the above transformation of R one can show that (27) is equivalent to the vanishing of the following expression

$$\widetilde{G}^a \bar{G}^b - \widetilde{G}^a (\bar{F} S_1)^b - (S_1 \widetilde{F})^a \bar{G}^b + (S_1 \widetilde{F})^a (\bar{F} S_1)^b + (S_2 \widetilde{F})^a (\bar{F} S_2)^b +$$
$$-i \left((S_2 \widetilde{F})^a \bar{G}^b - (S_2 \widetilde{F})^a (\bar{F} S_1)^b - \widetilde{G}^a (\bar{F} S_2)^b + (S_1 \widetilde{F})^a (\bar{F} S_2)^b \right) .$$

Using the relation $\mathcal{G} = R^{-\dagger}(G - S_1 F)$, which follows from (2) and (24), the first and second lines of this expression are equivalent to the left hand side of (25) and (26) respectively. Thus (27) is satisfied concluding the proof that the theory with the Lagrangian L is invariant under $U(n, n)$.

Conversely, if we are given a Lagrangian L with equations of motion invariant under $U(n, n)$ we can obtain a theory without the scalar field S by setting $S = i$. Then the duality group is broken to the stability group of $S = i$ which is $U(n) \times U(n)$, the maximal compact subgroup. Thus we can easily move between the theory with a scalar field S and the theory without S.

[b]Note that S_2 is a positive definite hermitian metric and R is a vielbein. The Lagrangian only depends on the metric and the arbitrariness in the choice of vielbein introduces a $U(n)$ gauge invariance.

We also give the infinitesimal transformation of $\mathcal{F}$ and $\mathcal{G}$

$$\begin{aligned} \delta\mathcal{G} &= RCR^\dagger\mathcal{G} - iRCR^\dagger\mathcal{F} \ , \\ \delta\mathcal{F} &= -RCR^\dagger\mathcal{F} - iRCR^\dagger\mathcal{G} \ . \end{aligned} \tag{28}$$

The last term in (28) is a unitary transformation and could be canceled by using a different choice for the transformation of R. The first term is an infinitesimal duality transformation belonging to the maximal compact subgroup $U(n) \times U(n)$. Note however that it is a space-time dependent duality transformation.

Next we find the differential equation that a Lagrangian must satisfy if the equations of motion are invariant under the maximal compact duality group. We are therefore considering a Lagrangian without the scalar field S. We will also assume that the auxiliary fields have been eliminated, the field strengths appear in the Lagrangian only through the Lorentz invariant combinations

$$\alpha^{ab} \equiv \frac{1}{2}F^a\bar{F}^b, \quad \beta^{ab} \equiv \frac{1}{2}\widetilde{F}^a\bar{F}^b, \tag{29}$$

and that the Lagrangian is a sum of traces (or of products of traces) of monomials in α and β. If the Lagrangian has such a form, equation (23) is satisfied. Then under a compact duality transformation the variation of the Lagrangian is

$$\delta\mathcal{L} = \mathrm{Tr}(\mathcal{L}_\alpha\delta\alpha + \mathcal{L}_\beta\delta\beta) \ ,$$

where we define

$$\mathcal{L}_\alpha \equiv \frac{\partial\mathcal{L}}{\partial\alpha^T} \ , \quad \mathcal{L}_\beta \equiv \frac{\partial\mathcal{L}}{\partial\beta^T} \ .$$

Using the definitions (2) and (29), we find that (22) is equivalent to

$$\mathcal{L}_\beta\beta\mathcal{L}_\beta - \mathcal{L}_\alpha\beta\mathcal{L}_\alpha + \mathcal{L}_\alpha\alpha\mathcal{L}_\beta + \mathcal{L}_\beta\alpha\mathcal{L}_\alpha + \beta = 0 \ . \tag{30}$$

This is a generalization of the differential equation introduced in [10] where the case of a single real gauge field was considered. Equation (30) is invariant under the following transformation

$$\begin{aligned} \alpha' &= \alpha \ , \\ \beta' &= -\beta \ . \end{aligned} \tag{31}$$

If one considers a self-dual theory with n real field strengths F_R, where now α and β are defined by $\alpha^{ab} = 1/4\ F_R^a F_R^b$ and $\beta^{ab} = 1/4\ F_R^a \tilde{F}_R^b$, equation (30) still holds. In this case one can extend the duality group from $U(n)$ to $Sp(2n, \mathbb{R})$ by introducing scalar fields as in [21]. Although these remarks will be central in later arguments, their proofs closely resemble those in the case of complex fields, so we omit them.

3 Born-Infeld with Auxiliary Fields

In this section we describe a $U(n,n)$ duality invariant nonlinear gauge theory with n complex gauge fields [11]. The use of auxiliary fields in the Lagrangian is inspired by the work of [12, 13] and simplifies the check of duality invariance.

We begin with the following Lagrangian introduced in [11]

$$L = \mathrm{Re}\,\mathrm{Tr}\,[\, i(\lambda - S)\chi - \frac{i}{2}\lambda\chi S_2\chi^\dagger + i\lambda\mathcal{N}\,]\ , \tag{32}$$

where $\mathcal{N} = \alpha - i\beta$. As mentioned in Section 2, here we do not require S to be symmetric. The auxiliary fields χ and λ are n dimensional complex matrices. If we could solve their equations of motion and use the solution in the Lagrangian (32) we would find a Lagrangian which depends only on α, β and S. Obtaining this Lagrangian is the main thrust of our paper.

If we set $S = i$ in the above Lagrangian, the theory is only self-dual under the maximal compact subgroup $U(n) \times U(n)$, as discussed in Section 2. However, if we now reintroduce the scalar field as in (17), the new Lagrangian is the same as (32) only after field redefinitions of χ and λ. We can also add a kinetic term for the scalar field S. This term must be duality invariant since, as we will see shortly, the rest of the Lagrangian already satisfies the self-duality condition (11). For example we can add a nonlinear σ-model Lagrangian defined on the coset space $U(n,n)/U(n) \times U(n)$ with the metric given by

$$\mathrm{Tr}\left[(S^\dagger - S)^{-1} dS^\dagger (S - S^\dagger)^{-1} dS\right]. \tag{33}$$

The metric (33) is Kähler since it is obtained from the Kähler potential

$$K = \mathrm{Tr}\ln(S_2)\ . \tag{34}$$

This Kähler potential changes by a Kähler transformation under (20); this ensures that the metric is duality invariant.

It will be convenient to decompose the auxiliary fields into hermitian matrices, as we have already done for S,

$$S = S_1 + iS_2\ ,\quad \lambda = \lambda_1 + i\lambda_2\ ,\quad \chi = \chi_1 + i\chi_2\ .$$

To prove the duality of (32) we first note that the last term in the Lagrangian can be written as

$$\mathrm{Re}\,\mathrm{Tr}\,[\,i\lambda(\alpha - i\beta)\,] = \mathrm{Tr}(-\lambda_2\alpha + \lambda_1\beta)\ .$$

If the field λ transforms by fractional transformation and the λ_i's and the gauge fields are real this is the $U(1)^n$ Maxwell action, with the gauge fields interacting with the scalar field λ, and this term by itself has the correct transformation properties under the duality group [3]. Similarly for hermitian α, β and λ_i this term by itself satisfies equation (11). It follows that the rest of the Lagrangian must be duality invariant. The duality transformations of the scalar and auxiliary fields are

$$S' = (aS+b)(cS+d)^{-1}, \tag{35}$$

$$\lambda' = (a\lambda+b)(c\lambda+d)^{-1}, \tag{36}$$

$$\chi' = (c\lambda+d)\chi(cS^\dagger+d)^\dagger. \tag{37}$$

To show the invariance of $\mathrm{Tr}[i(\lambda - S)\chi]$ it is convenient to rewrite (35) as

$$S' = (cS^\dagger+d)^{-\dagger}(aS^\dagger+b)^\dagger.$$

The proof of invariance of the remaining term which can be written as

$$\mathrm{Re}\,\mathrm{Tr}\,[-\frac{i}{2}\lambda\chi S_2\chi^\dagger] = \mathrm{Tr}\,[\frac{1}{2}\lambda_2\chi S_2\chi^\dagger]\ ,$$

is straightforward using the following transformations obtained from (35), (36) and (37)

$$\begin{aligned} S_2' &= (cS^\dagger + d)^{-\dagger} S_2 (cS^\dagger + d)^{-1}, \\ \lambda_2' &= (c\lambda + d)^{-\dagger} \lambda_2 (c\lambda + d)^{-1}, \\ \chi'^\dagger &= (cS^\dagger + d)\chi^\dagger (c\lambda + d)^\dagger \ . \end{aligned} \tag{38}$$

The Lagrangian has also a discrete parity symmetry which acts on the fields as

$$\begin{aligned} \alpha' &= \bar{\alpha} \ , \\ \beta' &= -\bar{\beta} \ , \\ S' &= -\bar{S} \ , \\ \chi' &= \bar{\chi} \ , \\ \lambda' &= -\bar{\lambda} \ . \end{aligned} \tag{39}$$

Although the theory of duality invariance presented in the previous section guarantees that this theory is self-dual, one can also check directly that the equations of motion obtained by varying the auxiliary fields are preserved under duality rotations. These equations of motion are

$$L_\lambda \equiv \frac{\partial L}{\partial \lambda^T} = i(\chi - \frac{1}{2}\chi S_2 \chi^\dagger + \alpha - i\beta) = 0 \ , \tag{40}$$

$$L_\chi \equiv \frac{\partial L}{\partial \chi^T} = i(\lambda - S - iS_2 \chi^\dagger \lambda_2) = 0 \ , \tag{41}$$

and indeed these two equations form a multiplet under duality transformations. Using the explicit forms (40) and (41) one can check that

$$\begin{aligned} \delta L_\lambda &= (C\lambda + D)L_\lambda + L_\lambda(\lambda C + D^\dagger) + \chi L_\chi C \ , \\ \delta L_\chi &= -(SC + D^\dagger)L_\chi - L_\chi(C\lambda + D) \ . \end{aligned}$$

Alternatively, one can obtain these equations directly from (13).

4 Elimination of the Auxiliary Fields

In this section we study the equation of motion (40) and attempt to solve for χ. We conjecture the form the Lagrangian assumes after the elimination of the auxiliary fields. This form is a generalization of the well-known Born-Infeld Lagrangian to more than one gauge field.

Using the equation of motion (40) in the Lagrangian (32) we obtain

$$L = \mathrm{Re}\,\mathrm{Tr}\,[-iS\chi] = \mathrm{Tr}\,[\,S_2\chi_1 + S_1\chi_2\,]\ , \tag{42}$$

where χ is now a function of α, β and S_2 that solves (40). For $n = 1$ we have to solve a second order algebraic equation and we obtain

$$\chi = \frac{1 - \sqrt{1 + 2S_2\alpha - S_2^2\beta^2}}{S_2} + i\beta\ .$$

Apart from the fact that the gauge fields are complex the result is the Born-Infeld Lagrangian

$$L = 1 - \sqrt{1 + 2S_2\alpha - S_2^2\beta^2} + S_1\beta\ . \tag{43}$$

In fact, for $n = 1$ we could have taken the gauge fields to be real even in the formulation with auxiliary fields as in [13], in which case the duality group becomes the $Sp(2,\mathbb{R})$ subgroup of $U(1,1)$ obtained by requiring a, b, c and d to satisfy (16) and to be real.

We now study equation (40) for arbitrary n. First notice that (40) can be simplified with the following field redefinitions

$$\begin{aligned} \widehat{\chi} &= R\chi R^\dagger\ , \\ \widehat{\alpha} &= R\alpha R^\dagger\ , \\ \widehat{\beta} &= R\beta R^\dagger\ , \end{aligned} \tag{44}$$

where, as in (18), $R^\dagger R = S_2$. The equation of motion for χ is then equivalent to

$$\widehat{\chi} - \frac{1}{2}\widehat{\chi}\widehat{\chi}^\dagger + \widehat{\alpha} - i\widehat{\beta} = 0\ . \tag{45}$$

Breaking this equation into its hermitian and antihermitian parts we find

$$\widehat{\chi}_2 = \widehat{\beta} \ , \tag{46}$$

$$\widehat{\chi}_1 = \frac{1}{2}(\widehat{\chi}_1^2 - 2\widehat{\alpha} + \widehat{\beta}^2 + i[\widehat{\beta}, \widehat{\chi}_1]) \ . \tag{47}$$

It is convenient to define

$$X = 1 - \widehat{\chi}_1 \ .$$

Then (47) is equivalent to the quadratic equation for the hermitian matrix X

$$X^2 = 1 + 2\widehat{\alpha} - \widehat{\beta}^2 + i[\widehat{\beta}, X] \ . \tag{48}$$

In terms of X the Lagrangian (32) takes the form

$$L = \mathrm{Tr}\,[\,1 - X \ + S_1\beta] \ , \tag{49}$$

where here X is now a function of $\widehat{\alpha}$ and $\widehat{\beta}$ that satisfies (48).

For $n = 1$ the equation (48) can be solved trivially since it is a second order algebraic equation. For arbitrary n, it becomes a matrix equations whose closed form solution does not seem to be known. However we solved for X as a power series in $\widehat{\alpha}$ and $\widehat{\beta}$

$$X = \sum_{m\geq 0} \frac{1}{m!} X^m \ . \tag{50}$$

Here m refers to the combined power of $\widehat{\alpha}$ and $\widehat{\beta}$ in each term. Then X can be solved perturbatively using the recursion relation obtained from (48)

$$X^0 = 1 \ , \quad X^1 = \widehat{\alpha} \ , \quad X^2 = -\widehat{\alpha}^2 - \widehat{\beta}^2 + i[\widehat{\beta}, \widehat{\alpha}] \ ,$$

$$\forall\ m > 2 \ , \quad 2X^m = -\sum_{j=1}^{m-1} \binom{m}{j} X^j X^{m-j} + i\,m\,[\widehat{\beta}, X^{m-1}] \ . \tag{51}$$

The initial condition for the recursion relation $X^0 = 1$ guarantees that the Lagrangian has a physical weak field limit. We have not been able to solve explicitly the recursion relation (51) and obtain X to all orders. However, to obtain the action only $\mathrm{Tr}\,[X]$ is needed.

It was conjectured in [11] and checked up to the sixth order that inserting the solution of (48) into (32) gives the following Lagrangian

$$L = \mathrm{Tr}\,[\,1 - \mathcal{S}_{\widehat{\alpha},\widehat{\beta}}\sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2} + S_1\beta\,]\ . \tag{52}$$

The square root is to be understood in terms of its power series expansion. The symmetrizer $\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}$ acts by symmetrizing each monomial with respect to the $\widehat{\alpha}$ and $\widehat{\beta}$ variables[c], and is normalized so that $\mathcal{S}_{\widehat{\alpha},\widehat{\beta}} \circ \mathcal{S}_{\widehat{\alpha},\widehat{\beta}} = \mathcal{S}_{\widehat{\alpha},\widehat{\beta}}$. It is a linear operator which maps a monomial of order m in $\widehat{\alpha}$ and $\widehat{\beta}$ into $1/m!$ times the polynomial obtained by summing all $m!$ permutations of the monomial. Let $P_{r,s}(\widehat{\alpha},\widehat{\beta})$ be the symmetric polynomial of order r in $\widehat{\alpha}$ and of order s in $\widehat{\beta}$. It is the sum of all the $\binom{r+s}{r}$ different words of length $r+s$ for which r of the letters are $\widehat{\alpha}$ and s of the letters are $\widehat{\beta}$. We can write the following explicit formula for $P_{r,s}(\widehat{\alpha},\widehat{\beta})$

$$P_{r,s}(\widehat{\alpha},\widehat{\beta}) = \frac{1}{r!s!}\left(\frac{\partial}{\partial\mu}\right)^r\left(\frac{\partial}{\partial\nu}\right)^s(\mu\widehat{\alpha} + \nu\widehat{\beta})^{r+s}\ . \tag{53}$$

Here ν and μ are commuting variables. Let M^{rs} be an arbitrary monomial with unit coefficient of order r in $\widehat{\alpha}$ and of order s in $\widehat{\beta}$. Then the symmetrizer acts on M^{rs} as follows

$$\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}(M^{rs}) = \begin{pmatrix} r+s \\ r \end{pmatrix}^{-1} P_{r,s}(\widehat{\alpha},\widehat{\beta})\ . \tag{54}$$

The explicit form of the symmetrized square root term appearing in the Lagrangian is given by

$$\mathcal{S}_{\widehat{\alpha},\widehat{\beta}}\sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2} \;=\; \sum_{r,s\geq 0}(-1)^{(r+1)}\ \frac{(2r+2s-3)!!\ (2s)!}{2^s\ (r+2s)!\ s!}\ P_{r,2s}(\widehat{\alpha},\widehat{\beta})\ ,$$

where $(-3)!! = -1\,,\ (-1)!! = 1\,.$

[c]To avoid confusion, we remark here that the nonabelian Born-Infeld Lagrangian introduced in [22, 23] also involves a symmetrized trace. However, while in [22, 23] the symmetrization is in the nonabelian field strength here the symmetrization is in $\widehat{\alpha}$ and $\widehat{\beta}$.

The conjecture (52) can be sharpened by stating that the solution $X = X(\widehat{\alpha}, \widehat{\beta})$ of equation (48) satisfies

$$\mathcal{C}\,(X) = \mathcal{S}_{\widehat{\alpha},\widehat{\beta}} \sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2}\ , \tag{55}$$

where $\mathcal{C}$ is the cyclic average operator defined as follows: $\mathcal{C}$ is a linear operator and it maps a monomial of order m in $\widehat{\alpha}$ and $\widehat{\beta}$ to $1/m$ times the polynomial obtained by summing all m cyclic permutations of the monomial. The normalization of $\mathcal{C}$ guarantees that $\mathcal{C} \circ \mathcal{C} = \mathcal{C}$ and also $\mathrm{Tr} \circ \mathcal{C} = \mathrm{Tr}$. Notice that the nontrivial statement in (55) is $\mathcal{C}(X) = \mathcal{S}_{\widehat{\alpha},\widehat{\beta}}(X)$, that is the cyclic average of X is a completely symmetrized quantity. Using the sharper conjecture (55) it is straighforward to see that the Lagrangian (49)

$$L = \mathrm{Tr}\,[\,1 - X + S_1\beta\,] = \mathrm{Tr}\,[\,1 - \mathcal{C}(X) + S_1\beta\,]$$

takes the form (52). We have no general analytic proof of (55) but we have checked it up to order seventeen for arbitrary noncommuting variables $\widehat{\alpha}$ and $\widehat{\beta}$ using the Mathematica computer program. In the Appendix we present an alternative expansion, which is equivalent to (51), and which is more convenient for checking the conjecture (55) order by order by hand and by computer.

5 Real field Strengths

We now show that our results imply the existence of a Born-Infeld theory with n real field strengths which is duality invariant under the maximal duality group $Sp(2n, \mathbb{R})$.

We first study the case without scalar fields, i.e. $S_1 = 0$ and $S_2 = R = 1$. Consider a Lagrangian $\mathcal{L} = \mathcal{L}(\alpha, \beta)$ which describes a self-dual theory with complex gauge fields. We will assume that the Lagrangian is a sum of traces (or of products of traces) of monomials in α and β. It follows that this Lagrangian satisfies the self-duality equations (30). This remains true in the special case that α and β are real. That is $\mathcal{L} = \mathcal{L}(\alpha, \beta)$ satisfies the self-duality equation (30) with $\alpha = \alpha^T = \bar{\alpha}$ and $\beta = \beta^T = \bar{\beta}$. We now recall

that equation (30) is also the self-duality condition for Lagrangians with real gauge fields provided that α and β are defined in the following way

$$\alpha^{ab} = \frac{1}{4} F_R^a F_R^b \ , \quad \beta^{ab} = \frac{1}{4} \widetilde{F}_R^a F_R^b \ , \tag{56}$$

where F_R^a denotes a real field strength. This implies that the theory described by the Lagrangian $\mathcal{L}_R = \mathcal{L}(\alpha(F_R^a), \beta(F_R^a))$ is self-dual with duality group $U(n)$, the maximal compact subgroup of $Sp(2n, \mathbb{R})$. The duality group can be extended to the full noncompact $Sp(2n, \mathbb{R})$, the maximal duality group of n real field strengths [3], by introducing the scalar fields S via the prescription (17) which also applies to the real case provided S is symmetric [21].

In our case the Lagrangian $L = \mathrm{Tr}\,[1 - X(\widehat{\alpha}, \widehat{\beta}) + S_1 \beta]$, where $X(\widehat{\alpha}, \widehat{\beta})$ is the solution of (48), defines a duality invariant theory because it is obtained from the Lagrangian with auxiliary fields (32) that is explicitly self-dual. Therefore $L_R = \mathrm{Tr}\,[1 - X(\widehat{\alpha}, \widehat{\beta}) + S_1 \beta]$ with the field strengths taken real is also self-dual. Using the conjecture (55) we obtain an explicit formula for the Born-Infeld Lagrangian with real gauge fields describing an $Sp(2n, \mathbb{R})$ duality invariant theory

$$L_R = \mathrm{Tr}\ [\, 1 - \mathcal{S}_{\widehat{\alpha}, \widehat{\beta}} \sqrt{1 + 2\widehat{\alpha} - \widehat{\beta}^2} + S_1 \beta \,] \ .$$

6 Supersymmetric Theory

In this section we briefly discuss supersymmetric versions of some of the Lagrangians introduced. First we discuss the supersymmetric form of the Lagrangian (32). Consider the superfields $V^a = \frac{1}{\sqrt{2}}(V_1^a + iV_2^a)$ and $\check{V}^a = \frac{1}{\sqrt{2}}(V_1^a - iV_2^a)$ where V_1^a and V_2^a are real vector superfields, and define

$$W_\alpha^a = -\frac{1}{4} \bar{D}^2 D_\alpha V^a \ , \quad \check{W}_\alpha^a = -\frac{1}{4} \bar{D}^2 D_\alpha \check{V}^a \ .$$

Both W^a and $\check{W}^a$ are chiral superfields and can be used to construct a matrix of chiral superfields

$$\mathcal{M}^{ab} \equiv W^a \check{W}^b \ .$$

The supersymmetric version of the Lagrangian (32) is then given by

$$L = \mathrm{Re} \int d^2\theta \left[\mathrm{Tr}\,(i(\lambda - S)\chi - \frac{i}{2}\lambda \bar{D}^2(\chi S_2 \chi^\dagger) - i\lambda \mathcal{M}) \right] \ , \qquad (57)$$

where S, λ and χ denote chiral superfields with the same symmetry properties as their corresponding bosonic fields. While the bosonic fields S and λ appearing in (32) are the lowest component of the superfields denoted by the same letter, the field χ in the action (32) is the highest component of the superfield χ. A supersymmetric kinetic term for the scalar field S can be written using the Kähler potential (34) as described in [24].

Just as in the bosonic Born-Infeld, one would like to eliminate the auxiliary fields. However we have not been able to do this exactly except for $n = 1$, and unlike the bosonic case we do not even have a conjectured form of the Lagrangian without auxiliary fields. For $n = 1$ just as in the bosonic case the theory with auxiliary fields also admits both a real and a complex version, i.e. we can also consider a Lagrangian with a single real superfield. Then we can integrate out the auxiliary superfields and obtain the supersymmetric version of (52)

$$L = \int d^4\theta \frac{S_2^2 W^2 \bar{W}^2}{1 - A + \sqrt{1 - 2A + B^2}} + \mathrm{Re} \left[\int d^2\theta (-\frac{i}{2} S W^2) \right] \ , \qquad (58)$$

where

$$A = \frac{1}{4}(D^2(S_2 W^2) + \bar{D}^2(S_2 \bar{W}^2)) \ , \quad B = \frac{1}{4}(D^2(S_2 W^2) - \bar{D}^2(S_2 \bar{W}^2)) \ .$$

If we only want a $U(1)$ duality invariance we can set $S = i$ and then the action (58) reduces to the supersymmetric Born-Infeld action described in [14, 15, 16].

In the case of weak fields the first term of (58) can be neglected and the Lagrangian is quadratic in the field strengths. Under these conditions the combined requirements of supersymmetry and self duality can be used [25] to constrain the form of the weak coupling limit of the effective Lagrangian from string theory.

7 Extension to Arbitrary Even Dimensions

In a space-time of arbitrary even dimension, $D = 2p$ we define the matrices

$$\alpha^{ab} = \frac{1}{p!}\, F^a_{\mu_1\ldots\mu_p} \bar{F}^{b\,\mu_1\ldots\mu_p} \ , \quad \beta^{ab} = \frac{1}{p!}\, \tilde{F}^a_{\mu_1\ldots\mu_p} \bar{F}^{b\,\mu_1\ldots\mu_p} \ , \tag{59}$$

where $\tilde{F}^a_{\mu_1\ldots\mu_p} = 1/p!\ \varepsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} F^{a\,\nu_1\ldots\nu_p}$ is the Hodge dual of F^a. The dual field strength is given by

$$\tilde{G}^a_{\mu_1\ldots\mu_p} \equiv p!\, \frac{\partial L}{\partial \bar{F}^{a\,\mu_1\ldots\mu_p}} \ , \quad \tilde{\tilde{G}}^a_{\mu_1\ldots\mu_p} \equiv p!\, \frac{\partial L}{\partial F^{a\,\mu_1\ldots\mu_p}} \ .$$

Since $\tilde{\tilde{F}} = (-1)^{p+1} F$ and $\tilde{F} G = (-1)^p F \tilde{G}$, for all even dimensions the matrix α is hermitian, while β is hermitian if $D = 4\nu$ and anti-hermitian if $D = 4\nu + 2$. It is also convenient to define

$$\mathcal{N} = \begin{cases} \alpha - i\beta \ , & \text{if} \quad D = 4\nu \ , \\ \alpha + \beta \ , & \text{if} \quad D = 4\nu + 2 \ . \end{cases}$$

With these definitions the Lagrangian (32) gives a $U(n,n)$ duality invariant theory in arbitrary even dimensions.

However, if the dimension of space-time is $D = 4\nu + 2$, where ν is integer it is convenient to make the following field redefinitions

$$\Lambda = i\lambda \ , \quad \mathcal{S} = iS \ .$$

The new fields have the decomposition

$$\Lambda = -\Lambda_1 + \Lambda_2 \ , \quad \mathcal{S} = -\mathcal{S}_1 + \mathcal{S}_2 \ ,$$

where Λ_1 and $\mathcal{S}_1$ are hermitian and Λ_2 and $\mathcal{S}_2$ are anti-hermitian. The minus sign was introduced so that we have

$$\mathcal{S}_1 = S_2 \ . \tag{60}$$

Then $\mathcal{S}_1$ is positive definite and we can write $\mathcal{S}_1 = R^\dagger R$ with R an arbitrary nonsingular n-dimensional matrix.

We also perform a similarity transformation on the $U(n,n)$ duality group, such that the transformation properties of the new fields simplify. Let us define two $2n$-dimensional matrices with the block form

$$\mathbb{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \mathbb{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

and let the matrices T and M have the block decomposition

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad T = \begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} \ .$$

Then one can define the $U(n,n)$ group as the group of matrices satisfying either one of the two relations

$$M\,\mathbb{K}\,M^\dagger = \mathbb{K} \ , \quad T\,\mathbb{H}\,T^\dagger = \mathbb{H} \ . \tag{61}$$

The two definitions are related by a unitary transformation $M = \mathcal{U}^{-1}T\mathcal{U}$ where

$$\mathcal{U} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} . \tag{62}$$

The n-dimensional matrices $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ satisfy

$$\underline{a}^\dagger\underline{d} + \underline{c}^\dagger\underline{b} = 1 \ , \quad \underline{c}^\dagger\underline{a} + \underline{a}^\dagger\underline{c} = 0 \ , \quad \underline{b}^\dagger\underline{d} + \underline{d}^\dagger\underline{b} = 0 \ . \tag{63}$$

The action of $U(n,n)$ on the scalar fields is given by

$$\begin{aligned} \mathcal{S}' &= (\underline{a}\mathcal{S} + \underline{b})(\underline{c}\mathcal{S} + \underline{d})^{-1} \ , \\ \Lambda' &= (\underline{a}\Lambda + \underline{b})(\underline{c}\Lambda + \underline{d})^{-1} \ , \\ \chi' &= (\underline{c}\Lambda + \underline{d})\chi(-\underline{c}\mathcal{S}^\dagger + \underline{d})^\dagger \ . \end{aligned} \tag{64}$$

Note that the positivity of $\mathcal{S}_1$ is compatible with the above transformation law of $\mathcal{S}$.

The Lagrangian, written in terms of the redefined fields, takes the form

$$L = \mathrm{Re}\left[\mathrm{Tr}\left((\Lambda - \mathcal{S})\chi - \frac{1}{2}\Lambda\chi\mathcal{S}_1\chi^\dagger + \Lambda\mathcal{N}\right)\right] . \tag{65}$$

Our conjecture regarding the Lagrangian without auxiliary fields is independent of the dimension of space-time and if it holds we can eliminate the auxiliary fields to obtain the Lagrangian

$$L = \mathrm{Tr}\,[\,1 - \mathcal{S}_{\widehat{\alpha},\widehat{\beta}}\sqrt{1 + 2\widehat{\alpha} + \widehat{\beta}^2} + \mathcal{S}_2\beta\,]\ , \tag{66}$$

where

$$\begin{aligned} \widehat{\alpha} &= R\alpha R^\dagger\ , \\ \widehat{\beta} &= R\beta R^\dagger\ . \end{aligned} \tag{67}$$

Note also that $\mathcal{S}_2$ appears in the last term of the Lagrangian (66), and this is consistent with $\mathcal{S}_2$ and β being anti-hermitian in space-times of odd half dimension. Also, there is a change of sign in front of the $\hat{\beta}^2$ term under the square root in (66) due to the change in the definition of $\mathcal{N}$.

If the half-dimension of space-time is odd it is consistent to take all the fields to be real in either the Lagrangian with auxiliary fields (65), or in the Lagrangian (66) where the auxiliary fields have been eliminated. Then we obtain a theory invariant under an $O(n,n)$ duality group. It was shown in [17, 21] that the maximal connected duality group for a theory of dimension $D = 4\nu + 2$ with n antisymmetric tensors is $SO(n,n)$. In the analysis of [17, 21] only infinitesimal duality transformations were considered, and from these one can only show duality under the connected component of the group. In [19, 20] the group $O(n,n)$ was considered. Note that, as discussed in Appendix B, $O(n,n)$ has four disjoint components embedded in $U(n,n)$ which is a connected group. Finally, one can also obtain a theory invariant under the $O(n) \times O(n)$ maximal compact subgroup of $O(n,n)$ by setting $\mathcal{S} = -1$ in the Lagrangian (66).

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Appendix A

In this appendix we discuss an equivalent expansion of the Lagrangian (42) which simplifies the order by order check of the conjecture. We set $S = i$ for simplicity, since S can always be reintroduced via the prescription (17). The expansion is in terms of the variables p and q defined as

$$p \equiv -\frac{1}{2}(\alpha - i\beta) \ , \quad q \equiv -\frac{1}{2}(\alpha + i\beta) \ . \tag{68}$$

Note that the self-duality equation (30) simplifies when written in terms of p and q

$$p - \mathcal{L}_p \, p \, \mathcal{L}_p = q - \mathcal{L}_q \, q \, \mathcal{L}_q \ . \tag{69}$$

Next we describe the perturbative expression of X. Let us define

$$\chi = 2P \ , \quad \chi^\dagger = 2Q \ .$$

Then the equations of motion (40) for χ and its hermitian conjugate become

$$P = PQ + p \ , \quad Q = PQ + q \ .$$

It is convenient to consider the following expansions

$$P = \sum_n P_n \ , \quad Q = \sum_n Q_n \ ,$$

we then have

$$P_0 = Q_0 = 0 \ , \quad P_1 = p \ , \quad Q_1 = q$$

and we can solve for P_n recursively

$$P_n = \sum_{r=1}^{n-1} P_r Q_{n-r} \ . \tag{70}$$

Notice that since $\chi - \chi^\dagger = 2i\beta$, for $n > 1$, $P_n = Q_n$. Therefore we have $P_0 = 0$, $P_1 = p$, $P_2 = pq$ and, for all $n > 2$,

$$P_n = pP_{n-1} + P_{n-1}q + \sum_{r=2}^{n-2} P_r P_{n-r} \ . \tag{71}$$

We also have $P_n = -\frac{1}{2\,n!} X^n$ for all $n > 1$. The Lagrangian is now expressed as

$$\mathcal{L} = \mathrm{Re}\,\mathrm{Tr}\,\chi = \mathrm{Re}\,\mathrm{Tr}\,[\,2\sum_n P_n] = \mathrm{Tr}\,[\,p + q + 2\sum_{n\geq 2} P_n\,] \ .$$

Using (53) and the linear change of variables (68) one can prove that symmetrization with respect to α and β is equivalent to symmetrization with respect to p and q. Then we can rewrite the conjectured symmetrized square root Lagrangian as

$$\mathcal{L} = \mathrm{Tr}[1 - \mathcal{S}_{p,q}\sqrt{1 - 2(p+q) + (p-q)^2}\,] \ . \tag{72}$$

We believe the explicit power series expansion of the square root in p and q has the simple expression

$$\mathcal{S}_{p,q}\sqrt{1 - 2(p+q) + (p-q)^2} = 1 - p - q - 2\sum_{r,s\geq 1} \frac{1}{r+s}\binom{r+s-2}{r-1} P_{r,s}(p,q) \ ,$$

which has been checked up to order twenty in p and q with the Mathematica computer program. Using the above expansion we can rewrite the conjecture (55) in the p and q variables

$$\mathcal{C}\left(1 - p - q - 2\sum_{n\geq 2} P_n\right) = 1 - p - q - 2\sum_{r,s\geq 1} \frac{1}{r+s}\binom{r+s-2}{r-1} P_{r,s}(p,q) \ .$$

It is this form that has been checked up to order seventeen by computer.

Appendix B

In this appendix we show that the field S provides a global parametrization of the coset space G/H where G is $U(n,n)$, $Sp(2n,\mathbb{R})$ or $O(n,n)$ and H is the maximal compact subgroup of G. We will concentrate on $U(n,n)$ but the same argument applies for the other groups.

Cosets are equivalence classes of group elements g of G under right multiplication with arbitrary elements h of H

$$g \sim gh \ .$$

We denote the coset containing g by gH. The maximal compact subgroup of G is defined as

$$H \equiv \{h \in G \,|\, hh^\dagger = h^\dagger h = 1\} \ .$$

It is the intersection of $U(n,n)$ with $U(2n)$ i.e. $U(n) \times U(n)$.

Next consider the map $\phi : G/H \to C$ defined by

$$\phi(gH) = gg^\dagger \ ,$$

where

$$C = \{s \in G \,|\, s^\dagger = s, \ s \text{ positive definite}\}$$

is the subset of hermitian positive definite group elements of G. This map is well defined since for any two elements g and g' in the same coset, $g' = gh$ and $g'g'^\dagger = ghh^\dagger g^\dagger = gg^\dagger$. Furthermore this map is one to one. We show first that the map is surjective. Let s be an arbitrary hermitian positive definite element of G. Then

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-\dagger} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} \ . \tag{73}$$

The last equality in (73) can be checked using the group relations (16). The decomposition exists whenever d is invertible, but since s is positive definite and d is the restriction of s on an n-dimensional subspace d is also positive

definite. Note also that $d^\dagger = d$ and $(bd^{-1})^\dagger = bd^{-1} = d^{-1}c$. Then g defined as

$$g = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1/2} & \\ 0 & d^{1/2} \end{pmatrix}$$

satisfies $s = gg^\dagger$, thus the map ϕ is surjective. To show that the map is also injective note that $gg^\dagger = g'g'^\dagger$ is equivalent to $g'^{-1}g(g'^{-1}g)^\dagger = 1$. Then $h = g'^{-1}g$ is an element of G satisfying $hh^\dagger = 1$, that is it belongs to the maximal compact subgroup H and we have $g = g'h$ so g and g' belong to the same coset.

If we define $S_2 = d^{-1}$ and $S_1 = bd^{-1}$ we can rewrite (73) as

$$s = \begin{pmatrix} 1 & S_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_2 & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ S_1 & 1 \end{pmatrix} .$$

This decomposition can also be written in terms of S_2 and $S = S_1 + iS_2$ as

$$s = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & S^\dagger \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_2^{-1} & 0 \\ 0 & S_2^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ S & 1 \end{pmatrix} . \tag{74}$$

Left multiplication on the group G induces an action of the group G on the coset space

$$s' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} s \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger .$$

Using the decomposition (74) one can easily show that the fractional transformation (19) of S is equivalent to this action. The form (74) is very convenient since the first term is invariant under the action, while the second term only contains S and S_2 and these have the simple transformation properties (19) and (21).

If we make all the matrices above real we obtain the parametrizations of $Sp(2n, \mathbb{R})/U(n)$. If we change the basis with the unitary matrix $\mathcal{U}$ defined in (62) and then require all the matrices to be real we obtain the coset space $O(n,n)/O(n) \times O(n)$.

Since the map ϕ is injective we see that S, such that S_2 is positive definite, is a global coordinate on the coset space $U(n,n)/U(n) \times U(n)$. Thus

this coset space is connected. The group $U(n,n)$ is a principal bundle over $U(n,n)/U(n)\times U(n)$ with a $U(n)\times U(n)$ fiber. The number of disconnected components of a principal bundle with a connected base is at most equal to the number of components of the fiber which in this case is one. Thus $U(n,n)$ is connected. Using the same argument one can show that $Sp(2n,\mathbb{R})$ is connected while $O(n,n)$ has at most four components. By an argument similar to the one used for the Lorentz group one can show that there are at least four components. Thus, as mentioned in Section 7, $O(n,n)$ has exactly four components.

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