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Strings in AdS_3 and the $SL(2,R)$ WZW Model. Part 1: The Spectrum

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Abstract

In this paper we study the spectrum of bosonic string theory on AdS_3. We study classical solutions of the $SL(2,R)$ WZW model, including solutions for long strings with non-zero winding number. We show that the model has a symmetry relating string configurations with different winding numbers. We then study the Hilbert space of the WZW model, including all states related by the above symmetry. This leads to a precise description of long strings. We prove a no-ghost theorem for all the representations that are involved and discuss the scattering of the long string.

[1] On leave of absence from the University of California, Berkeley.

1 Introduction

In this paper we study the spectrum of critical bosonic string theory on $AdS_3 \times \mathcal{M}$ with NS-NS backgrounds, where $\mathcal{M}$ is a compact space. Understanding string theory on AdS_3 is interesting from the point of view of the AdS/CFT correspondence since it enables us to study the correspondence beyond the gravity approximation. Another motivation is to understand string theory on a curved space-time, where the timelike component g_{00} of the metric is non-trivial.

This involves understanding the $SL(2,R)$ WZW model. In this paper, we always consider the case when the target space is the universal cover of the $SL(2,R)$ group manifold so that the timelike direction is non-compact. The states of the WZW model form representations of the current algebras $\widehat{SL}(2,R)_L \times \widehat{SL}(2,R)_R$. Once we know which representations of these algebras appear, we can find the physical states of a string in AdS_3 by imposing the Virasoro constraints on the representation spaces. The problem is to find the set of representations that one should consider. In WZW models for compact groups, the unitarity restricts the possible representations [1]. Representations of $\widehat{SL}(2,R)$, on the other hand, are not unitary except for the trivial representation. Of course this is not a surprise; the physical requirement is that states should have non-negative norms only after we impose the Virasoro constraints. Previous work on the subject [2, 3, 4, 5, 6, 7, 8, 9, 10] typically considered representations with L_0 bounded below and concluded that the physical spectrum does not contain negative norm states if there is the restriction $0 < j < k/2$ on the $SL(2,R)$ spin j of the representation; the spin of the $SL(2,R)$ is roughly the mass of the string state in AdS_3.

This restriction raises two puzzles. One is that it seems to imply an upper bound on the mass of the string states in AdS_3 so that the internal energy of the string could not be too high. For example, if the compact space $\mathcal{M}$ has a nontrivial 1-cycle, we find that there is an upper bound on the winding number on the cycle. This restriction, which is independent of the string coupling, looks very arbitrary and raises doubts about the consistency of the theory. The second puzzle is that, on physical grounds, we expect that the theory contains states corresponding to the long strings of [11, 12]. These are finite energy states where we have a long string stretched close to the boundary of AdS_3. These states are not found in any representation with L_0 bounded below. In this paper, we propose that the Hilbert space of the WZW model includes a new type of representations, and we show that this proposal resolves both the puzzles. In these new representations, L_0 is not bounded below. They are obtained by acting on the standard representations by elements of the loop group that are not continuously connected to the identity, through an operation called spectral flow. These representations in the $SL(2,R)$ WZW model have also been considered, with some minor variations, in

[13, 14]. The authors of these papers were motivated by finding a modular invariant partition function. They were, however, considering the case when the target space is $SL(2,R)$ group manifold and not its universal cover.

Throughout this paper, we consider AdS_3 in global coordinates, which do not have a coordinate horizon. In these coordinates, the unitarity issue becomes clearer since strings cannot fall behind any horizon. The interested reader could refer to [15, 16, 17] for studies involving AdS_3 in Poincare coordinates. From the point of view of the AdS/CFT correspondence, it is the spectrum of strings on AdS_3 in the global coordinates that determines the spectrum of conformal dimensions of operators in the boundary CFT, though in principle the same information could be extracted from the theory in Poincare coordinates.

In order to completely settle the question of consistency of the $SL(2,R)$ WZW model, one needs to show that the OPE of two elements of the set of representations that we consider contains only elements of this set. We plan to discuss this issue in our future publication.

The organization of this paper is as follows. In section 2, we study classical solutions of the $SL(2,R)$ WZW model and we show that the model has a spectral flow symmetry which relates various solutions. In section 3, we do a semi-classical analysis and have the first glimpse of what happens when we raises the internal excitation of the string beyond the upper bound implied by the restriction $j < k/2$. In section 4, we study the full quantum problem and we propose a set of representations that gives a spectrum for the model with the correct semi-classical limits. In section 5, we briefly discuss scattering amplitudes involving the long strings. We conclude the paper with a summary of our results in section 6. In appendix A, we extend the proof of the no-ghost theorem for the representations we introduced in section 4. In appendix B, we study the one-loop partition function in AdS_3 with the Lorentzian signature metric and show how the sum over spectral flow reproduces the result [18] after taking an Euclidean signature metric, up to contact terms in the modular parameters of the worldsheet.

2 Classical solutions

We start by choosing a parameterization of the $SL(2,R)$ group element as

$$\begin{aligned} g &= e^{iu\sigma_2} e^{\rho\sigma_3} e^{iv\sigma_2} \\ &= \begin{pmatrix} \cos t \cosh\rho + \cos\phi \sinh\rho & \sin t \cosh\rho - \sin\phi \sinh\rho \\ -\sin t \cosh\rho - \sin\phi \sinh\rho & \cos t \cosh\rho - \cos\phi \sinh\rho \end{pmatrix}. \end{aligned} \tag{1}$$

Here σ^i $(i = 1, 2, 3)$ are the Pauli matrices[2], and we set

$$u = \frac{1}{2}(t + \phi) \ , \qquad v = \frac{1}{2}(t - \phi). \tag{2}$$

Another useful parameterization of g is

$$g = \begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & X_{-1} - X_1 \end{pmatrix}, \tag{3}$$

with

$$X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 = 1. \tag{4}$$

This parameterization shows that the $SL(2, R)$ group manifold is a 3-dimensional hyperboloid. The metric on AdS_3,

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2,$$

is expressed in the global coordinates (t, ϕ, ρ) as

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2. \tag{5}$$

We will always work on the universal cover of the hyperboloid (4), and t is non-compact.

Our theory has the WZW action

$$S = \frac{k}{8\pi\alpha'} \int d^2\sigma \mathrm{Tr} \left(g^{-1}\partial g g^{-1}\partial g \right) + k\Gamma_{WZ} \tag{6}$$

The level k is not quantized since H^3 vanishes for $SL(2, R)$. The semi-classical limit corresponds to large k. We define the right and left moving coordinates on the world-sheet as,

$$x^{\pm} = \tau \pm \sigma, \tag{7}$$

where σ is periodic with the period 2π. This action has a set of conserved right and left moving currents

$$J_R^a(x^+) = k\mathrm{Tr} \left(T^a \partial_+ g g^{-1} \right), \qquad J_L^a(x^-) = k\mathrm{Tr} \left(T^{a*} g^{-1} \partial_- g \right) \tag{8}$$

where T^a are a basis for the $SL(2, R)$ Lie algebra. It is convenient to take them as

$$T^3 = -\frac{i}{2}\sigma^2, \quad T^{\pm} = \frac{1}{2}(\sigma^3 \pm i\sigma_1).$$

[2] $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In terms of our parameterization, the currents are expressed as

$$\begin{aligned} J_R^3 =& k(\partial_+ u + \cosh 2\rho \partial_+ v) \\ J_R^\pm =& k(\partial_+ \rho \pm i \sinh 2\rho \partial_+ v) e^{\mp i 2u} \ , \end{aligned} \tag{9}$$

and

$$\begin{aligned} J_L^3 =& k(\partial_- v + \cosh 2\rho \partial_- u) \\ J_L^\pm =& k(\partial_- \rho \pm i \sinh 2\rho \partial_- u) e^{\mp i 2v} \ . \end{aligned} \tag{10}$$

The zero modes of $J^3_{R,L}$ are related to the energy E and angular momentum ℓ in AdS_3 as

$$\begin{aligned} J_0^3 &= \int_0^{2\pi} \frac{dx^+}{2\pi} J_R^3 = \frac{1}{2}(E+\ell) \\ \bar{J}_0^3 &= \int_0^{2\pi} \frac{dx^-}{2\pi} J_L^3 = \frac{1}{2}(E-\ell). \end{aligned} \tag{11}$$

The second Casimir of $SL(2,R)$ is

$$c_2 = J^a J^a = \frac{1}{2}\left(J^+ J^- + J^- J^+\right) - (J^3)^2 \ . \tag{12}$$

The equations of motion derived from (6) is $\partial_-(\partial_+ g g^{-1}) = 0$, namely that the currents, J_R and J_L, are purely right or left moving as indicated. A general solution of the equations of motion for $SL(2,R)$ is the product of two group elements each of which depends only on x^+ or x^- as

$$g = g_+(x^+) g_-(x^-). \tag{13}$$

Comparing (13) with (1) we can find the embedding of the worldsheet in AdS_3. The requirement that the string is closed under $\sigma \to \sigma + 2\pi$ imposes the constraint,

$$g_+(x^+ + 2\pi) = g_+(x^+) M, \qquad g_-(x^- - 2\pi) = M^{-1} g_-(x^-), \tag{14}$$

with the same $M \in SL(2,R)$ for both g_+ and g_-. The monodromy matrix M is only defined up to a conjugation by $SL(2,R)$, and classical solutions of the WZW model are classified according to the conjugacy class of M.

For strings on $AdS_3 \times \mathcal{M}$, we should impose the Virasoro constraints

$$T_{++}^{total} = T_{++}^{AdS} + T_{++}^{other} = 0 \tag{15}$$

and similarly $T_{--}^{total} = 0$, where

$$T_{++}^{AdS} = \frac{1}{k} J_R^a J_R^a$$

is the energy-momentum tensor for the AdS_3 part[3] and T_{++}^{other} represents the energy-momentum tensor for the sigma-model on $\mathcal{M}$.

Let us analyze some simple classical solutions.

[3] In the quantum theory, we will have the same expression but with $k \to k-2$.

2.1 Geodesics in AdS_3

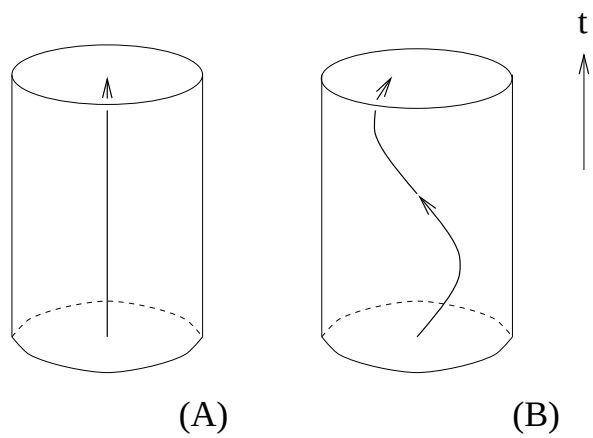

Figure 1: Timelike geodesic; (A) a solution (18) with $U = V = 1$, (B) a general geodesic is obtained by acting the $SL(2,R) \times SL(2,R)$ isometry on (A).

Consider a solution

$$g_+ = Ue^{iv_+(x^+)\sigma_2}, \quad g_- = e^{iu_-(x^-)\sigma_2}V, \tag{16}$$

where U and V are constant elements of $SL(2,R)$. The energy momentum tensor of this solution is

$$T^{AdS}_{++} = -k(\partial_+ v_+)^2, \;\; T^{AdS}_{--} = -k(\partial_- u_-)^2. \tag{17}$$

Suppose we have some string excitation in the compact part $\mathcal{M}$ of $AdS_3 \times \mathcal{M}$, and set $T^{other}_{\pm\pm} = h$ for some constant $h > 0$. We may regard h as a conformal weight of the sigma-model on $\mathcal{M}$. The Virasoro constraints $T^{total}_{\pm\pm} = 0$ implies

$$(\partial_+ v_+)^2 = (\partial_- u_-)^2 = \frac{h}{k}.$$

Thus we can set $v_+ = \alpha x^+/2$ and $u_- = \alpha x^-/2$ where $\alpha = \pm\sqrt{4h/k}$. Substituting this in (13), we obtain

$$g = U \begin{pmatrix} \cos(\alpha\tau) & \sin(\alpha\tau) \\ -\sin(\alpha\tau) & \cos(\alpha\tau) \end{pmatrix} V. \tag{18}$$

Since the solution depends only on τ and not on σ, we interpret that the string is collapsed to a point which flows along the trajectory in AdS_3 parameterized by τ. See Figure 1. If $U = V = 1$, the solution (18) represents a particle sitting at the center of AdS_3,

$$t = \alpha\tau, \quad \rho = 0. \tag{19}$$

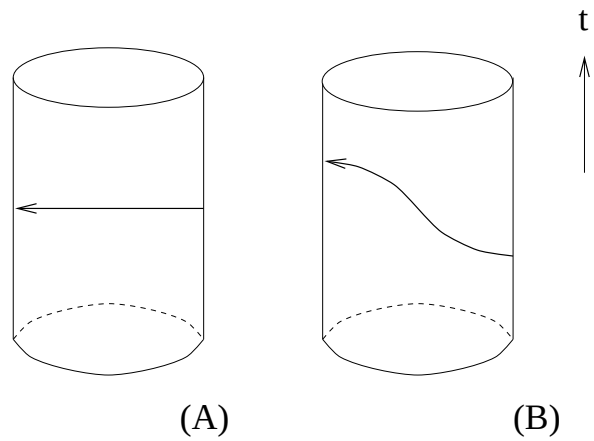

Figure 2: Spacelike geodesic; (A) a solution (21) with $U = V = 1$, (B) a general geodesic is obtained by acting the $SL(2,R) \times SL(2,R)$ isometry on (A).

A more general solution (18) is given by acting the $SL(2,R) \times SL(2,R)$ isometry on (19), and therefore it is a timelike geodesic[4] in AdS_3. For this solution, the currents are given by

$$J_R^a T^a = \frac{k}{2} \alpha U T^3 U^{-1}, \tag{20}$$

and similarly for J_L. The monodromy matrix M defined by (14) is

$$M = \begin{pmatrix} \cos(\alpha\pi) & \sin(\alpha\pi) \\ -\sin(\alpha\pi) & \cos(\alpha\pi) \end{pmatrix}$$

and belongs to the elliptic conjugacy class $SL(2,R)$.

A solution corresponding to a spacelike geodesic is

$$g = U \begin{pmatrix} e^{\alpha\tau} & 0 \\ 0 & e^{-\alpha\tau} \end{pmatrix} V, \tag{21}$$

with $U, V \in SL(2,R)$. The energy-momentum tensor has a sign opposite of (17)

$$T_{\pm\pm}^{AdS} = \frac{1}{4} k \alpha^2. \tag{22}$$

If we choose $U = V = 1$, the solution is simply a straight line cutting the spacelike section $t = 0$ of AdS_3 diagonally,

$$t = 0, \quad \rho e^{i\phi} = \alpha\tau, \tag{23}$$

[4]In fact, any timelike geodesic can be expressed in the form (18).

See Figure 2 (A). A general solution (21) is given from this by the action of the isometry, and therefore is a spacelike geodesic. The currents for this solution are

$$J_R^a T^a = \frac{k}{2}\alpha U T^1 U^{-1}, \tag{24}$$

and the monodromy matrix is

$$M = \begin{pmatrix} e^{\alpha\pi} & 0 \\ 0 & e^{-\alpha\pi} \end{pmatrix},$$

which belongs to the hyperbolic conjugacy class of $SL(2,R)$.

There is one more class of solutions whose monodromy matrices are in the parabolic conjugacy class of $SL(2,R)$. They correspond to null geodesics in AdS_3.

2.2 Spectral flow and strings with winding numbers

Given one classical solution $g = \tilde{g}_+\tilde{g}_-$, we can generate new solutions by the following operation,

$$g_+ = e^{i\frac{1}{2}w_R x^+\sigma_2}\tilde{g}_+ \qquad g_- = \tilde{g}_- e^{i\frac{1}{2}w_L x^-\sigma_2}. \tag{25}$$

Comparing this with the parameterization (1) of $g = g_+g_-$, we see that this operation amounts to

$$\begin{aligned} t &\to t + \frac{1}{2}(w_R + w_L)\tau + \frac{1}{2}(w_R - w_L)\sigma \\ \phi &\to \phi + \frac{1}{2}(w_R + w_L)\sigma + \frac{1}{2}(w_R - w_L)\tau. \end{aligned} \tag{26}$$

The periodicity of the string worldsheet, under $\sigma \to \sigma + 2\pi$, on the universal cover of $SL(2,R)$ requires[5] $w_R = w_L = w$ for some integer w.

One may regard (25) as an action by an element of the loop group $\widehat{SL}(2,R)\times\widehat{SL}(2,R)$ which is not continuously connected to the identity[6]. This particular symmetry of the theory will also be useful in our analysis of the Hilbert space. Here we see that it generates a new solution from an old solution. Furthermore, the currents (9) change in the following way

$$J_R^3 = \tilde{J}_R^3 + \frac{k}{2}w, \quad J_R^\pm = \tilde{J}_R^\pm e^{\mp iwx^+} \tag{27}$$

[5]If the target space is the single cover of $SL(2,R)$, w_R and w_L can be different. In this case $(w_R - w_L)$ gives the winding number along the closed timelike curve on $SL(2,R)$.

[6]The loop group $\widehat{SL}(2,R)$ has such an element since $\pi_1(SL(2,R)) = \mathbb{Z}$. Therefore, in the model whose the target space is the single cover of $SL(2,R)$, the full symmetry group of the model is the loop group of $SL(2,R)\times SL(2,R)$ and its connected components are parametrized by $\mathbb{Z}\times\mathbb{Z}$. In this paper, we are studying the model for the universal cover of $SL(2,R)$. In this case, some of these elements do not act properly on the field space, generating worldsheets which close only modulo time translation. However the ones parametrized by the diagonal $\mathbb{Z}$ are still symmetry of the model. The diagonal $\mathbb{Z}$ parameterizes the spectral flow operation performed simultaneously for both the left and right movers.

and a similar expression for J_L^a. Or, in terms of the Fourier modes,

$$J_n^3 = \tilde{J}_n^3 + \frac{k}{2} w \delta_{n,0}, \quad J_n^\pm = \tilde{J}_{n\mp w}^\pm. \tag{28}$$

This means that the stress tensor will change to

$$T_{++}^{AdS} = \tilde{T}_{++}^{AdS} - w\tilde{J}^3 - \frac{k}{4} w^2. \tag{29}$$

In the CFT literature, this operation is known as the spectral flow.

Let us study what happens if we act with this symmetry on the solutions corresponding to geodesics, (18) and (21). These solutions depend only on the worldsheet time coordinate τ, and the spectral flow (26) with $w = w_R = w_L$ introduces σ dependence as

$$\begin{aligned} t &= t_0(\tau) + w\tau \\ \rho &= \rho_0(\tau) \\ \phi &= \phi_0(\tau) + w\sigma. \end{aligned} \tag{30}$$

Here (t_0, ρ_0, ϕ_0) represents the original geodesic solution. So what the spectral flow does is to stretch the geodesic solution in the t-direction (by adding $w\tau$) and rotates it around w-times around the center $\rho = 0$ of AdS_3 (by adding $w\sigma$). It is clear that the resulting solution describes a circular string, winding w-times around the center of AdS_3. Since the spectral flow changes the energy-momentum tensor, we need to impose the physical state condition $T_{\pm\pm}^{AdS} + T_{\pm\pm}^{other} = 0$ with respect to the new energy-momentum tensor (29).

2.3 Short strings as spectral flow of timelike geodesics

A timelike geodesic in AdS_3 makes a periodic trajectory as shown in Figure 1, approaching the boundary of AdS_3, then coming back to the center and so on. In particular, when $V = U^{-1}$ in (18), the geodesic periodically passes through the center $\rho = 0$ of AdS_3, with the period 2π in the t-coordinate. The spectral flow,

$$t \to t + w\tau, \quad \phi \to \phi + w\sigma$$

stretches the geodesic in the time direction and rotate it around the center $\rho = 0$; it is pictorially clear that the resulting solution describes a circular string which repeats expansion and contraction. This is shown shown in Figure 3 in the case of $w = 1$. Assuming $T_{\pm\pm}^{other} = h$ as in the case of geodesics, the Virasoro constraint for the solution is

$$T_{++}^{total} = \tilde{T}_{++}^{AdS} - w\tilde{J}^3 - \frac{k}{4} w^2 + T_{++}^{other} = 0. \tag{31}$$

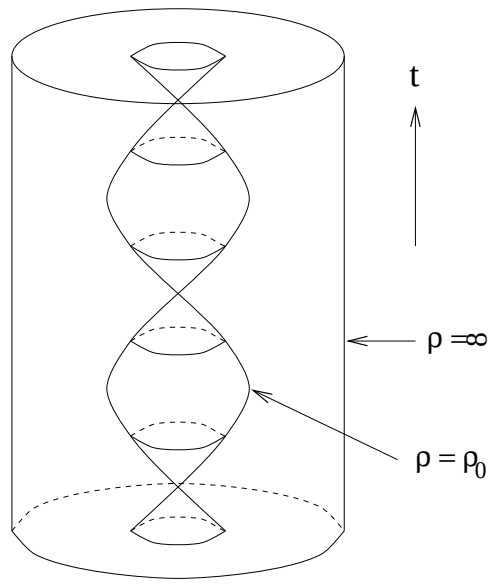

Figure 3: A classical solution obtained by the spectral flow of a timelike geodesic. The solution repeats expansion and contraction. The maximum size of the string is $\rho = \rho_0$.

Since

$$\tilde{T}^{AdS}_{++} = -\frac{k}{4}\alpha^2$$

for the timelike geodesic, we find

$$J^3_0 = \tilde{J}^3_0 + \frac{k}{2}w = \frac{k}{4}w + \frac{1}{w}\left(-\frac{k}{4}\alpha^2 + h\right). \tag{32}$$

The spacetime energy E of the string is given by $E = 2J^3_0$, and is bounded above as

$$E = \frac{k}{2}w + \frac{1}{w}\left(-k\alpha^2 + 2h\right) < \frac{k}{2}w + \frac{2h}{w}. \tag{33}$$

It is not difficult to find an explicit form of the solution. When $V = U^{-1}$ in (18), without loss of generality, we can set[7] $U = V^{-1} = e^{\frac{1}{2}\rho_0\sigma_3}$. The solution[8] obtained by the spectral flow of (18) is then

$$\begin{aligned} e^{i\phi}\sinh\rho &= ie^{iw\sigma}\sinh\rho_0\ \sin\alpha\tau \\ \tan t &= \frac{\tan w\tau + \tan\alpha\tau/\cosh\rho_0}{1 - \tan w\tau\tan\alpha\tau/\cosh\rho_0}. \end{aligned} \tag{34}$$

[7] A different choice of $U = V^{-1}$ simply results in shift of ϕ in the solution.

[8] We have been informed that a similar classical solution has also been studied in [19, 20].

The currents of this solution are

$$
\begin{aligned}
J_R^3 &= \frac{k}{2}(\alpha \cosh \rho_0 + w) \\
J_R^\pm &= \pm i \frac{k}{2} \alpha \sinh \rho_0 e^{\mp i w x^+},
\end{aligned}
\tag{35}
$$

and similarly for J_L. Comparing this with (32), we find

$$
\alpha = \alpha_\pm = -w \cosh \rho_0 \pm \sqrt{w^2 \sinh^2 \rho_0 + \frac{4h}{k}}. \tag{36}
$$

If we choose the branch $\alpha = \alpha_+$, the spacetime energy E of the solution is positive and is given by

$$
E = 2J_0^3 = 2\bar{J}_0^3 = k \left(\cosh \rho_0 \sqrt{\frac{4h}{k} + w^2 \sinh^2 \rho_0} - w \sinh^2 \rho_0 \right). \tag{37}
$$

There are several interesting features of this formula for the energy E. Except for the case of $h = kw^2/4$, the energy is a monotonically increasing function of ρ_0, which approaches $E \to kw/2 + 2h/w$ as $\rho_0 \to \infty$. One may view that the solution describe a bound state trapped inside of AdS_3. At the exceptional value of $h = kw^2/4$, we have $\alpha_+ = 0$ and the energy of the solution becomes $E = kw$, completely independent of the size ρ_0 of the string. The solution in this case is

$$
\rho = \rho_0, \quad t = w\tau, \quad \phi = w\sigma, \tag{38}
$$

and represents a string staying at the fixed radius $\rho = \rho_0$, neither contracting nor expanding. The fact that we have such a solution at any radius ρ_0 means that the string becomes marginally unstable in AdS_3.

Now let us turn to the case when $U \neq V^{-1}$, or to be more precise, when UV does not commute with $T^3 = -\frac{i}{2}\sigma^2$. (When UV commutes with T^3, one can shift the value of τ to set $U = V^{-1}$.) In this case, the geodesic does not necessarily pass through the center of AdS_3. Therefore the circular string obtained by its spectral flow does not collapse to a point. Since

$$
\tilde{J}^a{}_L T^{*a} = \frac{k}{2} \alpha U T^3 U^{-1}, \ \tilde{J}^a{}_R T^a = \frac{k}{2} \alpha V^{-1} T^3 V, \tag{39}
$$

$\tilde{J}_L^3 \neq \tilde{J}_R^3$ unless UV commutes with $T^3 = -\frac{i}{2}\sigma^2$, and the spacetime angular momentum $\ell = J_R^3 - J_L^3 = \tilde{J}_R^3 - \tilde{J}_L^3$ is nonzero. Thus one may view that the circular string is kept from completely collapsing by the centrifugal force. Since $T_{++}^{AdS} - T_{--}^{AdS} = -w(\tilde{J}_R^3 - \tilde{J}_L^3)$, the Virasoro constraint $T_{\pm\pm}^{total} = 0$ requires that the left and right conformal weights (h_L, h_R) of the internal part should be different and that $h_R - h_L = w\ell$.

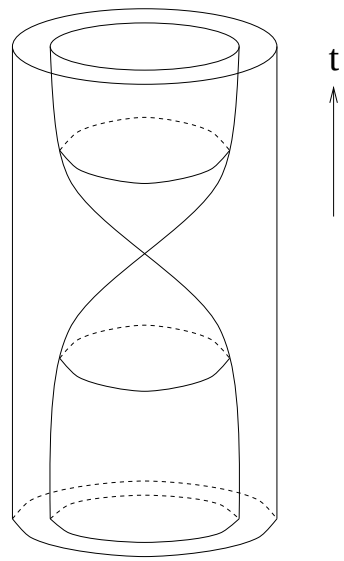

Figure 4: A long string solution obtained by the spectral flow of a spacelike geodesic. The long string comes from the boundary of AdS_3, collapse to a point and then expands away to the boundary of AdS_3 again.

2.4 Long strings as spectral flow of spacelike geodesics

We have seen in (33) that the spacetime energy E of the solution given by the spectral flow of the timelike geodesic is bounded above as $E < kw/2 + 2h/w$. What will happen if we raise the energy above this value? To understand this, let us look at the spectral flow of the spacelike geodesic. Since $\tilde{T}^{AdS}_{++} = +k\alpha^2/2$ for the spacelike geodesic, the Virasoro constraint (31) gives

$$J^3_0 = \tilde{J}^3_0 + \frac{k}{2}w = \frac{k}{4}w + \frac{1}{w}\left(\frac{k}{2}\alpha^2 + h\right), \tag{40}$$

and the spacetime energy is now bounded below,

$$E = 2J^3_0 > \frac{k}{2}w + \frac{2h}{w}. \tag{41}$$

As an example, let us consider the straight line cutting the spacelike section $t = 0$ diagonally (23). The spectral flow with w of this solution is

$$t = w\tau, \quad \rho e^{i\phi} = \alpha\tau e^{iw\sigma}, \tag{42}$$

namely

$$\rho = \frac{\alpha}{w}|t|. \tag{43}$$

The solution starts in the infinite past $t = -\infty$ as a circular string of an infinite radius located at the boundary of AdS_3. The string then collapse, shrinks to a point at $t = 0$, and expand away toward the boundary of AdS_3 as $t \to +\infty$. More generally, if we choose $U = V^{-1} = e^{-\frac{1}{2}\rho_0\sigma_1}$, the spectral flow of the geodesic (21) gives

$$\begin{aligned} e^{i\phi}\sinh\rho &= e^{iw\sigma}\cosh\rho_0\sinh\alpha\tau \\ \tan t &= \frac{\tan w\tau + \tanh\alpha\tau\sinh\rho_0}{1-\tan w\tau\tanh\alpha\tau\sinh\rho_0}. \end{aligned} \tag{44}$$

This solution, which we call a long string, is depicted in Figure 4.

The Virasoro constraint $T^{total}_{++} = 0$ for the long string (44) is

$$T^{AdS}_{++} + T^{other}_{++} = \frac{k}{4}\left(\alpha^2 - 2\alpha w\sinh\rho_0 - w^2\right) + h = 0, \tag{45}$$

with the solutions

$$\alpha = \alpha_\pm = w\sinh\rho_0 \mp \sqrt{w^2\cosh^2\rho_0 - \frac{4h}{k}}. \tag{46}$$

The spacetime energy E of these solutions are

$$E = 2J^2_0 = 2\bar{J}^3_0 = k\left(w\cosh^2\rho_0 \mp \sinh\rho_0\sqrt{w^2\cosh^2\rho_0 - \frac{4h}{k}}\right). \tag{47}$$

At the critical value $h = kw^2/4$, we have $\alpha_+ = 0$ and the energy for this solution becomes $E = kw$. At this point, the long string solution (44) coincides with (38). Thus we see that, as we increase the value of h to $h = kw^2/4$, the short string solution (34) can turn into the long string solution (44) and escape to infinity.

As explained in [11, 12], a string that winds in AdS_3 close to the boundary has finite energy because there is a balance between two large forces. One is the string tension that wants to make the string contract and the other is the NS-NS B field which wants to make the string expand. These forces cancel almost precisely near the boundary and only a finite energy piece is left. The threshold energy for the long string computed in [11, 12] is $kw/4$, in agreement with (41) when $h = 0$. These strings can have some momentum in the radial direction and that is a degree of freedom α that we saw explicitly above. One may view the long string as a scattering state, while the previous solution (34) is like a bound state trapped inside of AdS_3.

In general, if UV commutes with $T^3 = -\frac{i}{2}\sigma^2$, the long string collapses to a point once in its lifetime. If UV does not commute with T^3, the angular momentum $\ell = J^3_R - J^3_L$ of the solution does not vanish and the centrifugal force keeps the string from collapsing completely. In this case, the Virasoro constraint $T^{total}_{\pm\pm} = 0$ requires $h_R - h_L = w\ell$ for the conformal weights of the internal sector.

For the long strings, one can define a notion of the S-matrix. In the infinite past, the size of the long string is infinite but its energy is finite. Therefore the interactions between them are expected to be negligible, and one can define asymptotic states consisting of long strings. The strings then approach the center of AdS_3 and are scattered back to the boundary. In this process, the winding number could in principle change.

3 Semi-classical analysis

In studying the classical solutions, we were naively identifying the winding number w as associated to the cycle $\phi \to \phi + 2\pi$. But since this cycle is contractible in AdS_3, we should be careful about what we mean by the integer w. The winding number is well-defined when the string is close to the boundary, so we expect that long strings close to the boundary have definite winding numbers. On the other hand, when the string collapses to a point, as shown in Figures 3 and 4, the winding number is not well-defined. Therefore, if we quantize the string, it is possible to have a process in which the winding number changes. There is however a sense in which string states are characterized by some integer w.

In order to clarify the meaning of w when the string can collapse, let us look at the Nambu action

$$S = \int dt \frac{d\sigma}{2\pi} \left[\sqrt{det g_{ind}} - B_{t\phi} \partial_\sigma \phi \right] \tag{48}$$

where g_{ind} is the induced metric on the worldsheet, and $B_{t\phi}$ is the NS-NS B-field. We have chosen the static gauge in the time direction $t = \tau$. We assume that initially we have a state with $\rho = 0$, and we want to analyze small perturbations. Since the coordinate ϕ is not well-defined, it is more convenient to use

$$X^1 + iX^2 = \rho e^{i\phi}. \tag{49}$$

Let us compute the components of the induced metric g_{ind}. To be specific, we consider the case when the target space $AdS_3 \times S^3 \times T^4$, and consider a string winding around a cycle on T^4. By expanding in the quadratic order in ρ, we find

$$\begin{aligned} g_{ind,00} &= k \left[-(1+\rho^2) + \partial_0 X^a \partial_0 X^a \right] + \partial_0 Y^i \partial_0 Y^i \\ g_{ind,01} &= k \partial_0 X^a \partial_1 X^a + \partial_0 Y^i \partial_1 Y^i \\ g_{ind,11} &= k \partial_1 X^a \partial_1 X^a + \partial_1 Y^i \partial_1 Y^i, \quad (a = 1, 2) \end{aligned} \tag{50}$$

where Y^i's are coordinates on T^4. For simplicity, we consider purely winding modes on T^4, so that only $\partial_1 Y^i$ is nonzero. For these states, the conformal weight h is given

by[9]

$$4h = \oint \frac{d\sigma}{2\pi} G_{ij} \partial_1 Y^i \partial_1 Y^j \tag{51}$$

Substituting (50) and (51) into the action and expanding to the quadratic order in ρ, we find

$$\begin{aligned} S &= \sqrt{4kh} \int d\sigma^2 \left[1 - \frac{1}{2}(\partial_0 X^a)^2 + \frac{1}{2}\frac{k}{4h}\left(\partial_1 X^a + \epsilon_{ab}\sqrt{\frac{4h}{k}} X^b\right)^2 + \cdots \right] \\ &= \sqrt{4kh} \int d\sigma^2 \left[1 - \frac{1}{2}|\partial_0 \Phi|^2 + \frac{1}{2}\frac{k}{4h}\left|\left(\partial_1 - i\sqrt{\frac{4h}{k}}\right)\Phi\right|^2 + \cdots \right], \end{aligned} \tag{52}$$

where $\Phi = X^1 + iX^2$.

The action (52) is the one for a massless charged scalar field on $\mathbb{R} \times S^1$ coupled to a constant gauge field $A = \sqrt{4h/k}$ around S^1. As we vary A, we observe the well-know phenomenon of the spectral asymmetry. Let us first assume that A is not an integer. A general solution to the equation of motion derived from (52), requiring the periodicity in σ, is

$$\Phi \sim \sum_{n=-\infty}^{\infty} \left(a_n^\dagger e^{i(n-A)(\tilde{\tau}+\sigma)} + b_n e^{-i(n-A)(\tilde{\tau}-\sigma)} \right) \frac{e^{iA\sigma}}{n-A}, \tag{53}$$

where $A = \sqrt{4h/k}$ and $\tilde{\tau} = \tau/A$. Upon quantization, the commutation relations are given (modulo a positive constant factor) by

$$[a_n, a_m^\dagger] = (n-A)\delta_{n,m}, \quad [b_n, b_m^\dagger] = (n-A)\delta_{n,m}. \tag{54}$$

Notice that the sign in the right hand side of (54) determines whether a_n or $a_n^\dagger$ should be regarded as the annihilation operator. Thus, assuming that the Hilbert space is positive definite, the vacuum state is defined by

$$\begin{aligned} a_n|0\rangle = b_n|0\rangle = 0, \quad (n > A) \\ a_n^\dagger|0\rangle = b_n^\dagger|0\rangle = 0, \quad (n < A). \end{aligned} \tag{55}$$

For $\Phi = \rho e^{i\phi}$ given by (53) and $t = A\tilde{\tau}$, we find

$$\begin{aligned} J_R^+ &= k\left(e^{-it}\partial_+\Phi^* - \Phi^*\partial_+ e^{-it}\right) \sim -ik\sum_n a_n e^{-in(\tilde{\tau}+\sigma)} \\ J_R^- &= k\left(e^{it}\partial_+\Phi - \Phi\partial_+ e^{it}\right) \sim ik\sum_n a_n^\dagger e^{in(\tilde{\tau}+\sigma)}, \end{aligned} \tag{56}$$

[9] One factor of 2 comes from the fact that this includes left and right movers and the other from the fact that the expression for the energy involves $1/2Y'^2$.

and similarly for $J_L^{\pm}$. Therefore $J_n^+ = -ika_n$ and $J_n^- = ika_{-n}^\dagger$. The vacuum state $|0\rangle$ defined by (55) then obeys

$$J_n^+|0\rangle = 0 \quad (n > A), \quad J_n^-|0\rangle = 0 \quad (n > -A). \tag{57}$$

Thus the vacuum state $|0\rangle$ is not in a regular highest weight representation of the current algebra $\widehat{SL}(2,R)$. If we set

$$J_n^{\pm} = \tilde{J}_{n\mp w}^{\pm} \tag{58}$$

with the integer w defined by

$$w < A < w + 1, \tag{59}$$

then $|0\rangle$ obeys the regular highest weight condition with respect to $\tilde{J}_n^{\pm}$,

$$\tilde{J}_n^+|0\rangle = 0 \;\; (n \geq 1), \qquad \tilde{J}_n^-|0\rangle = 0 \;\; (n \geq 0) \tag{60}$$

The change of the basis (58) is nothing but the spectral flow (28) discussed earlier, so we can identify w as the amount of spectral flow needed to transform the string state into a string state which obeys the regular conditions (60). We have found that, for a given value of h, there is a unique integer of w associated to the string state. As we vary the conformal weight h, $A = \sqrt{4h/k}$ will become an integer. At that point, one of the modes of the field Φ will have a vanishing potential. In fact we can check that classically this potential is completely flat. Giving an expectation value to that mode, we find configurations as in (38). Corresponding to various values of its momentum in the radial direction, we have a continuum of states. So, at this value of h, we do not have a normalizable ground state; instead we have a continuum of states which are δ-function normalizable. If we continue to increase h, we find again normalizable states, but they are labeled by a new integer $(w+1)$. Notice that w is not directly related to the physical winding of the string. In fact by exciting a coherent state of the oscillators a_n or b_n we can find string states that look like expanding and collapsing strings with winding number n around the origin.

One of the puzzles we raised in the introduction was what happens when we increase the internal conformal weight h of the string beyond the upper bound implied by the restriction $j < k/2$ on the $SL(2,R)$ spin j due to the no-ghost theorem. In this section, we saw a semi-classical version of the puzzle and its resolution. When h reaches the bound, we find that the state can become a long string with no cost in energy. Above the bound, we should consider a Fock space with a different bose sea level. In the fully quantum description of the model given below, we will find a similar situation but with minor corrections.

4 Quantum string in AdS_3

The Hilbert space of the WZW model is a sum of products of representations of the left and the right-moving current algebras generated by

$$J_L^a = \sum_{n=-\infty}^{\infty} J_n^a e^{-inx^-}, \quad J_R^a = \sum_{n=-\infty}^{\infty} \bar{J}_n^a e^{-inx^+}, \tag{61}$$

with $a = 3, \pm$, obeying the commutation relations

$$\begin{aligned} [J_n^3, J_m^3] &= -\frac{k}{2} n \delta_{n+m,0} \\ [J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm \\ [J_n^+, J_m^-] &= -2J_{n+m}^3 + kn\delta_{n+m,0}, \end{aligned} \tag{62}$$

and the same for $\bar{J}_n^a$. We denote the current algebra by $\widehat{SL}_k(2,R)$. The Virasoro generator L_n are defined by

$$\begin{aligned} L_0 &= \frac{1}{k-2}\left[\frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2 + \sum_{m=1}^{\infty}(J_{-m}^+ J_m^- + J_{-m}^- J_m^+ - 2J_{-m}^3 J_m^3)\right] \\ L_{n\neq 0} &= \frac{1}{k-2}\sum_{m=1}^{\infty}(J_{n-m}^+ J_m^- + J_{n-m}^- J_m^+ - 2J_{n-m}^3 J_m^3) \end{aligned} \tag{63}$$

and obey the commutation relation

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{64}$$

where the central charge c is given by

$$c = \frac{3k}{k-2}. \tag{65}$$

We will find that the Hilbert space of the WZW model consists of sub-sectors parameterized by integer w, labeling the amount of spectral flow in a sense to be made precise below. We then formulate our proposal on how the complete Hilbert space of the WZW model is decomposed into representations of the current algebras and provide evidences for the proposal.

States in a representation of the current algebra are labeled by eigenvalues of L_0 and J_0^3. Since the kinetic term of the WZW model based on $SL(2,R)$ has an indefinite signature, it is possible that the Hilbert space of the model contains states with negative eigenvalues of L_0 as well as states with negative norms, and indeed both types of states appear as we will see below. For the moment, we will consider a representation in which eigenvalues of L_0 is bounded below. We call them *positive energy representations*, or

unflowed representations. Since the action of $J_n^{3,\pm}$ with $n \geq 1$ on a state lowers the eigenvalue of L_0 by n, there has to be a set of states which are annihilated by them. We will call such states the primary states of the positive energy representation. All other states in the representation are obtained by acting $J_{-n}^{3,\pm}$ $(n \geq 1)$ on the primary states. The ground states make a representation of $SL(2,R)$ generated by $J_0^{3,\pm}$. So let us review irreducible representations of $SL(2,R)$.

4.1 Representations of the zero modes

We expect that physical states of a string in AdS_3 have positive norms. Since $J_0^{3,\pm}$ commute with the Virasoro constraints, physical spectrum of the string must be in unitary representations of $SL(2,R)$. Most of the mathematical references on representation theory of $SL(2,R)$ deal with the case with compact time[10]; we are however interested in the case with non-compact time. A clear analysis from the algebraic point of view is presented in [22], which we now summarize with some minor changes is notation.

There are the following five types of unitary representations. All the representations are parameterized by j, which is related to the second Casimir $c_2 = \frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2$ as $c_2 = -j(j-1)$.

(1) Principal discrete representations (lowest weight):

A representation of this type is realized in the Hilbert space

$$\mathcal{D}_j^+ = \{|j;m\rangle : \ m = j, \ j+1, \ j+2, \cdots\},$$

where $|j;j\rangle$ is annihilated by J_0^- and $|j;m\rangle$ is an eigenstate of J_0^3 with $J_0^3 = m$. The representation is unitarity if j is real and $j > 0$. For representations of the group $SL(2,R)$, j is restricted to be a half of integer. Since we are considering the universal cover of $SL(2,R)$, j can be any positive real number.

(2) Principal discrete representations (highest weight):

A charge conjugation of (1). A representation of this type is realized in the Hilbert space

$$\mathcal{D}_j^- = \{|j;m\rangle : \ m = -j, \ -j-1, \ -j-2, \cdots\},$$

where $|j;j\rangle$ is annihilated by J_0^+ and $|j;m\rangle$ is an eigenstate of J_0^3 with $J_0^3 = m$. The representation is unitary if j is real and $j > 0$.

(3) Principal continuous representations:

A representation of this type is realized in the Hilbert space of

$$\mathcal{C}_j^\alpha = \{|j,\alpha;m\rangle : \ m = \alpha, \ \alpha \pm 1, \alpha \pm 2, \ \cdots\},$$

[10]For a review of representations of $SL(2,R)$, see for example [21].

where $|j,\alpha;m\rangle$ is an eigenstate of J_0^3 with $J_0^3 = m$. Without loss of generality, we can restrict $0 \leq \alpha < 1$. The representation is unitary if $j = 1/2 + is$ and s is real[11].

(4) Complementary representations:

A representation of this type is realized in the Hilbert space of

$$\mathcal{E}_j^\alpha = \{|j,\alpha;m\rangle : \ m = \alpha, \ \alpha \pm 1, \ \alpha \pm 2, \cdots\},$$

where $|j,\alpha;m\rangle$ is an eigenstate of J_0^3 with $J_0^3 = m$. Without loss of generality, we can restrict $0 \leq \alpha < 1$. The representation is unitary if j is real, with $1/2 < j < 1$ and $j - 1/2 < |\alpha - 1/2|$.

(5) Identity representation:

This is the trivial representation with $j = 0$.

The analysis that led to the above representation was completely algebraic and in a particular physical system we can have only a subset of all possible representations. Which of these representations appear in the Hilbert space of the WZW model? As the first approximation, let us consider the $k \to \infty$ limit. If we expand around a short string solutions, *i.e.* oscillations near geodesics in AdS_3, the WZW model in this limit reduces to the quantum mechanics on AdS_3. The Hilbert space of the quantum mechanical model is the space of square-integrable[12] functions $\mathcal{L}^2(AdS_3)$ on AdS_3. The isometry of AdS_3 is $SL(2,R) \times SL(2,R)$, and one can decompose $\mathcal{L}^2(AdS_3)$ into its unitary representations. It is convenient to choose the basis of the Hilbert space in the following way. For each representation $\mathcal{R}$, one can define a function on AdS_3 by $F_{m,\bar{m}}(g) = \langle m|g|\bar{m}\rangle$ where $g \in AdS_3$, *i.e.* universal cover of $SL(2,R)$, and $|m\rangle$ is an eigenstate of J_0^3 with $J_0^3 = m$. Thus, for a given representation $\mathcal{H}$ of $SL(2,R)$, the function $F_{m,\bar{m}}(g)$ on AdS_3 is in the tensor product of the representations $\mathcal{R} \times \mathcal{R}$ for the isometry group $SL(2,R) \times SL(2,R)$.

For a discrete representation $\mathcal{D}_j^\pm$, the wave-function $f(\rho)$ behaves as $f(\rho) \sim e^{-2j\rho}$ for large ρ. Thus $\phi \in \mathcal{L}^2(AdS_3)$ if $j > 1/2$. Notice that in the range $0 < j < 1$ we have two representations with the same value of the Casimir but only one is in $\mathcal{L}^2(AdS_3)$, the one with $1/2 < j < 1$. As explained in [23], one could modify the norm so that the second solution with $0 < j < 1/2$ becomes normalizable. This modification of the norm is j-dependent. Similarly, supplementary series representations need a j-dependent modification to the norm to render them normalizable [21]. Therefore these representations would appear in non-standard quantizations of geodesics, quantizations which do not use the $\mathcal{L}^2$ norm on AdS_3. In this paper, we will only consider the standard

[11] Strictly speaking the representation with $j = 1/2, \ \alpha = 1/2$ is reducible as the sum of a highest weight and a lowest weight representation with $j = 1/2$.

[12] Since AdS_3 is non-compact, we consider square-integrability in the delta-function sense.

quantization using the $\mathcal{L}^2$ norm for the zero modes[13]. Wave-functions in $\mathcal{C}^\alpha_{j=1/2+is}$ are also delta-function normalizable with respect to the $\mathcal{L}^2$ norm. It is known that $\mathcal{C}^\alpha_{j=1/2+is} \times \mathcal{C}^\alpha_{j=1/2+is}$ and $\mathcal{D}^\pm_j \times \mathcal{D}^\pm_j$ with $j > 1/2$ form the complete basis of $\mathcal{L}^2(AdS_3)$.

For discrete lowest weight representations, the second Casimir is bounded above as $c_2 = -j(j-1) \leq 1/4$. This corresponds to the well-known Breitenlohner-Freedman bound $(\text{mass})^2 \geq -1/4$ for the Klein-Gordon equation. For the principal continuous representation $\mathcal{C}^\alpha_j$ with $j = 1/2 + is$, the second Casimir is $c_2 = 1/4 + s^2$. Therefore an existence of such a particle would violate the Breitenlohner-Freedman bound. In the bosonic string theory, the only physical state of this type is the tachyon. In a perturbatively stable string theory, such particle states should be excluded from its physical spectrum. On the other hand, the continuous representations appear in $\mathcal{L}^2(AdS_3)$ and they are expected to be part of the Hilbert space of the WZW model before the Virasoro constraint is imposed.

4.2 Representations of the current algebra and no-ghost theorem

Given a unitary representation $\mathcal{H}$ of $SL(2,R)$, one can construct a representation of $\widehat{SL}(2,R)$ by regarding $\mathcal{H}$ as its primary states annihilated by $J^{3,\pm}_{n\geq 1}$. The full representation space is generated by acting $J^{3,\pm}_{n\leq -1}$ on $\mathcal{H}$. Following the discussion in the previous subsection, we consider the cases when $\mathcal{H} = \mathcal{C}^\alpha_{j=1/2+is}$ and $\mathcal{D}^\pm_j$ with $j > 1/2$. We denote by $\widehat{\mathcal{D}}^\pm_j$ and $\widehat{\mathcal{C}}^\alpha_j$ the representations of the full current algebra built on the corresponding representations of the zero modes. In Figure 5, we have shown the weight diagram of the positive energy representation $\widehat{\mathcal{D}}^+_j$.

A representation of $\widehat{SL}_k(2,R)$ in general contains states with negative norms. In order for a string theory on AdS_3 to be consistent, one should be able to remove these negative norm states by imposing the Virasoro constraint,

$$(L_n + \mathcal{L}_n - \delta_{n,0})|\text{physical}\rangle = 0, \quad n \geq 0, \tag{66}$$

on the Hilbert space for a single string state, where L_n is the Virasoro generator of the $SL(2,R)$ WZW model and $\mathcal{L}_n$ for the sigma-model on $\mathcal{M}$. It has been shown that this no-ghost theorem holds for states in $\widehat{\mathcal{C}}^\alpha_{j=1/2+is}$ or $\widehat{\mathcal{D}}^\pm_j$ with $0 < j < k/2$ [2, 3, 6, 7, 8, 9, 22].

The no-ghost theorem is proved by first showing that all the solutions to the Virasoro constraint (66) can be expressed, modulo null states, as states in the coset

[13]Notice however, that even if the primary states have $j > 1/2$, we could have states with smaller values of j_0 for the zero mode $SL(2,R)$ among the descendents, for example $J^-_{-1}|j\rangle$ with $1 < j < 3/2$, has $j_0 = j - 1 < 1/2$.

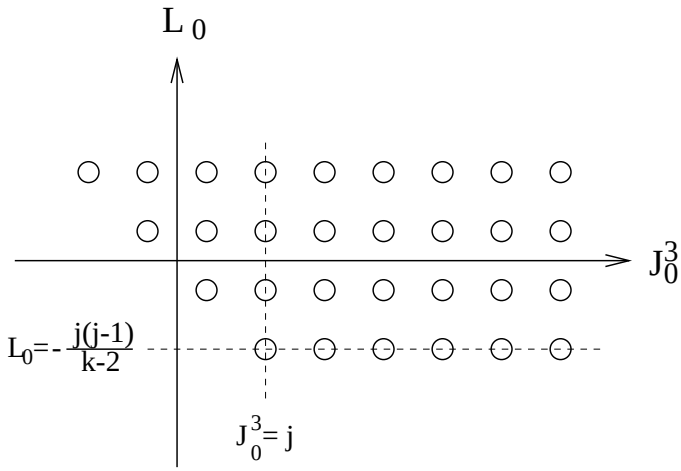

Figure 5: Weight diagram the representation $\widehat{\mathcal{D}}_j^+$, whose the primary states form a discrete lowest weight representation $\mathcal{D}_j^+$.

$SL(2,R)/U(1)$ obeying

$$J_n^3|\psi\rangle = 0 \ , \qquad n \geq 1. \tag{67}$$

This statement is true for $\widehat{\mathcal{C}}^{\alpha}_{1/2+is}$ and $\widehat{\mathcal{D}}_j^{\pm}$ with $0 < j < k/2$, if the total central charge of the Virasoro generator $L_n + \mathcal{L}_n$ is 26 [2, 3, 4, 6, 7, 8, 9] [14]. We review the proof of this statement in the appendix A.1. The second step is to show that the condition (67) removes all negative norm states. This was shown in [22] for the same class of representations.

The no-ghost theorem suggests that the spectrum of discrete representations has to be truncated for $j < k/2$. As we will see, this truncation is closely related to the existence of the long string states.

4.3 Spectral flow and the long string

The classical and semi-classical results discussed above indicate that, beyond positive energy representations that we have discussed so far, we have to include others related by spectral flow. To define a quantum version of the spectral flow, we note that, for any integer w, the transformation $J_n^{3,\pm} \to \tilde{J}_n^{3,\pm}$ given by

$$\tilde{J}_n^3 = J_n^3 - \frac{k}{2} w \delta_{n,0}, \quad \tilde{J}_n^+ = J_{n+w}^+, \quad \tilde{J}_n^- = J_{n-w}^-, \tag{68}$$

[14] We also assume $k > 2$.

preserves the commutation relations (62). The Virasoro generators $\tilde{L}_n$, which have the standard Sugawara form in terms of $\tilde{J}^a_n$, are different from L_n. They are given by

$$\tilde{L}_n = L_n + wJ^3_n - \frac{k}{4}w^2\delta_{n,0}. \tag{69}$$

Of course, they obey the Virasoro algebra with the same central charge c. This is the same formula as saw in the classical counterpart (29) of the spectral flow.

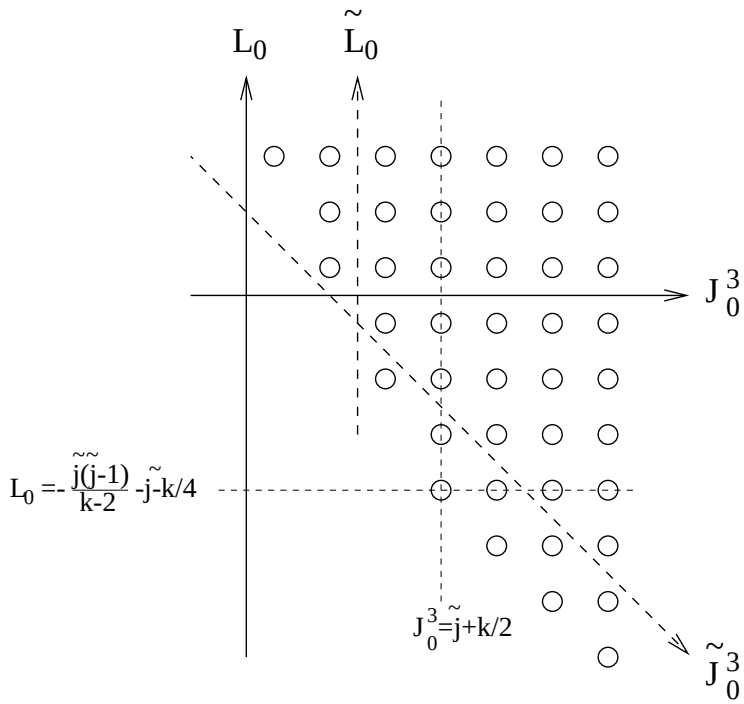

Figure 6: Weight diagram of the representation $\widehat{\mathcal{D}}^{+,w=1}_{\tilde{j}}$, which is the spectral flow of the diagram 5 with $w = 1$. The worldsheet energy L_0 of this representation is not bounded below, but the spacetime energy, J^3_0, is bounded below for states obeying the Virasoro constraint $L_0 = 1$.

The change of the basis (68) maps one representation into another, and this is called the spectral flow. In the case of a compact group such as $SU(2)$, the spectral flow maps a positive energy representation of the current algebra into another positive energy representation. An analogous transformation in the case of the $N = 2$ superconformal algebra in two dimensions has been used to construct the spacetime supercharges for superstring.

In the case of $SL(2, R)$, the spectral flow generates a new class of representations. As shown in Figure 6, the spectral flow with $w = 1$ maps the lowest weight representation

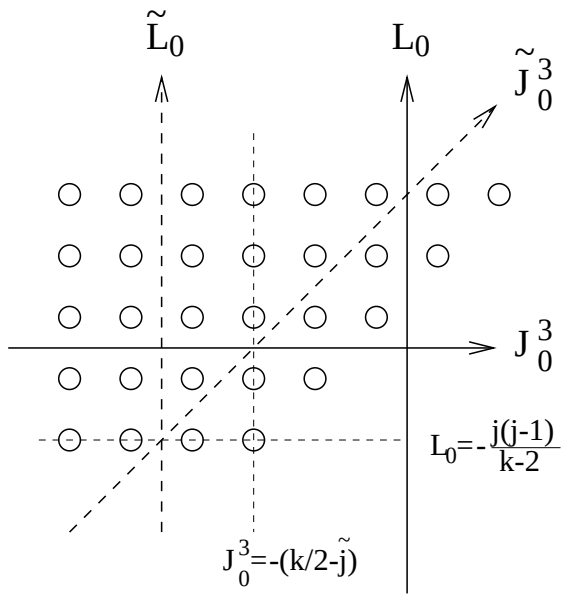

Figure 7: The spectral flow of the diagram 5 with $w=-1$. $\widehat{\mathcal{D}}^+_{\tilde{j}}$ is mapped to $\widehat{\mathcal{D}}^{+,w=-1}_{\tilde{j}} = \widehat{\mathcal{D}}^-_j$ with $j=k/2-\tilde{j}$. Since $\tilde{j}>1/2$, the resulting $\widehat{\mathcal{D}}^-_j$ obeys $j<(k-1)/2$. In particular, the unitarity bound $j<k/2$ required by the no-ghost theorem is satisfied.

$\widehat{\mathcal{D}}^+_{\tilde{j}}$ to a representation in which L_0 is not bounded below. The appearance of negative energy states is not too surprising since the kinetic term of the $SL(2,R)$ model is not positive definite. In general, a spectral flow of $\widehat{\mathcal{D}}^+_{\tilde{j}}$ with $w\geq 1$ or $w\leq -2$ gives a new representation in which L_0 is not bounded below. Similarly, the spectral flow of $\widehat{\mathcal{C}}^{\alpha}_{j=1/2+is}$ with $w\neq 0$ gives a representation in which L_0 is not bounded below. We denote the resulting representations by $\widehat{\mathcal{D}}^{\pm,w}_{\tilde{j}}$ and $\widehat{\mathcal{C}}^{\alpha,w}_{\tilde{j}}$, where $\tilde{j}$ labels the $SL(2,R)$ spin before the spectral flow.

These representations obtained by the spectral flow also contain negative norm states. In Appendix A.2, we generalize the proof of the no-ghost theorem and show that the Virasoro constraints indeed remove all negative norm states in the representations $\widehat{\mathcal{C}}^{\alpha,w}_{j=1/2+is}$ and $\widehat{\mathcal{D}}^{\pm,w}_{\tilde{j}}$ with $\tilde{j}<k/2$, for any integer w.

The only case where we get a representation with L_0 bounded below by the spectral flow is $\widehat{\mathcal{D}}^{\pm}_j$ with $w=\mp 1$. In this case, the representation is mapped to another positive energy representation $\widehat{\mathcal{D}}^{\pm,w=\mp 1}_{\tilde{j}} = \widehat{\mathcal{D}}^{\mp}_{k/2-\tilde{j}}$. Note that, if we start with the

representation with $\tilde{j} > 1/2$, the representation one gets after the spectral flow satisfies $j = k/2 - \tilde{j} < (k-1)/2$. Conversely, if there were a representation $\widehat{\mathcal{D}}_j^{\pm}$ with $j > (k-1)/2$ in the Hilbert space, the spectral flow would generate a representation $\widehat{\mathcal{D}}_j^{\mp}$ with $j < 1/2$, in contradiction with the standard harmonic analysis of the zero modes in section 4.1. Therefore, if we assume that the spectral flow is a symmetry of the WZW model, the discrete representations $\widehat{\mathcal{D}}_j^{\pm}$ appearing in the Hilbert space are automatically restricted to be in $1/2 < j < (k-1)/2$. In particular, the spectrum of j is truncated below the unitarity bound $j < k/2$ required by the no-ghost theorem. This further restriction on j was discussed in a related context by [24].

4.4 Physical spectrum

Let us consider first the spectrum for strings with $w = 0$. This is fairly standard. We start from an arbitrary descendent at level N in the current algebra and some operator of the internal CFT with conformal weight h. The L_0 constraint reads

$$(L_0 - 1)|j, m, N, h\rangle = 0 \Longrightarrow -\frac{j(j-1)}{k-2} + N + h - 1 = 0 \tag{70}$$

If we demand that $1/2 \leq j \leq (k-1)/2$, this equation will have a solution as long as $N + h$ is within the range

$$0 \leq N + h - 1 + \frac{1}{4(k-2)} \leq \frac{(k-2)}{4} \tag{71}$$

If we allow j to go all the way to $k/2$ we get $k/4$ on the right hand side of (71).

To analyze physical states of strings with $w \neq 0$, we start with a positive energy representation $\widehat{\mathcal{D}}_{\tilde{j}}^{+}$. After the spectral flow (68), a primary state $|\tilde{j}, \tilde{m}\rangle$ of $\widehat{\mathcal{D}}_{\tilde{j}}^{+}$, as a state of $\widehat{\mathcal{D}}_{\tilde{j}}^{+,w}$, obeys

$$\begin{aligned} &J^+_{n+w}|\tilde{j}, \tilde{m}\rangle = 0, \quad J^-_{n-w}|\tilde{j}, \tilde{m}\rangle = 0, \quad J^3_n|\tilde{j}, \tilde{m}\rangle = 0, \qquad n \geq 1 \\ &J^3_0|\tilde{j}, \tilde{m}\rangle = \left(\frac{k}{2}w + \tilde{m}\right)|\tilde{j}, \tilde{m}\rangle. \end{aligned} \tag{72}$$

Let us look for physical states with respect to the Virasoro generator L_n. From (72), we find the Virasoro constraints are

$$\begin{aligned} &(L_0 - 1)|\tilde{j}, \tilde{m}\rangle = \left(-\frac{\tilde{j}(\tilde{j}-1)}{k-2} - w\tilde{m} - \frac{k}{4}w^2 + \tilde{N} + h - 1\right)|\tilde{j}, \tilde{m}, \tilde{N}, h\rangle = 0 \\ &L_n|\tilde{j}, \tilde{m}\rangle = (\tilde{L}_n - w\tilde{J}^3_n)|\tilde{j}, \tilde{m}\rangle = 0, \qquad n \geq 1. \end{aligned} \tag{73}$$

where h is the contribution to the conformal weight from the internal CFT and $\tilde{N}$ is the level inside the current algebra before we take the spectral flow. The state obeys

the physical state conditions provided

$$\tilde{m} = -\frac{k}{4}w + \frac{1}{w}\left(-\frac{\tilde{j}(\tilde{j}-1)}{(k-2)} + \tilde{N} + h - 1\right). \tag{74}$$

The spacetime energy of this state measured by J_0^3 is then

$$J_0^3 = \tilde{m} + \frac{k}{2}w = \frac{k}{4}w + \frac{1}{w}\left(-\frac{\tilde{j}(\tilde{j}-1)}{(k-2)} + \tilde{N} + h - 1\right). \tag{75}$$

This is the quantum version of the classical formula (32), with the replacement

$$\frac{k}{4}\alpha^2 \to \frac{\tilde{j}(\tilde{j}-1)}{k-2} + 1.$$

Notice that $\tilde{m} = \tilde{j} + q$ where q is some integer, which could be negative[15]. Therefore the physical state condition becomes

$$\tilde{j} = \frac{1}{2} - \frac{k-2}{2}w + \sqrt{\frac{1}{4} + (k-2)\left(h - 1 + N_w - \frac{1}{2}w(w+1)\right)}. \tag{76}$$

Here

$$N_w = \tilde{N} - wq \tag{77}$$

is the level of the current algebra after the spectral flow by the amount w. Notice that the equation for $\tilde{j}$ is invariant under $\tilde{N} \to \tilde{N} \pm w$, $q \to q \pm 1$. This is reflecting the fact that $J_0^{\pm} = \tilde{J}^{\pm}_{\mp w}$ commute with the Virasoro constraints and generate the spacetime $SL(2,R)$ multiplets. In particular, we see that the spacetime $SL(2,R)$ representations that we get are lowest energy representations, since repeated action of $J_0^- = \tilde{J}_w^-$ will eventually annihilate the state. In fact, it is shown in Appendix A.2 that the only physical state with zero spacetime energy, $J_0^3 = 0$, is the state $J_{-1}^-|j=1\rangle$, and its complex conjugate. This physical state corresponds to the dilaton field in AdS_3, which played an important role in the analysis of the spacetime Virasoro algebra in [25]. All other states (except the tachyon with $w = 0$) have nonzero energy, and form highest/lowest weight representations of $SL(2,R)$ spacetime algebra. The negative energy ones are the complex conjugates of the positive energy ones.

By solving the on-shell condition (76) for $\tilde{j} > 0$ and substituting it into (75), one finds that the spacetime energy of the string is given by

$$\frac{E+\ell}{2} = J_0^3 = q + w + \frac{1}{2} + \sqrt{\frac{1}{4} + (k-2)\left(h - 1 + N_w - \frac{1}{2}w(w+1)\right)}. \tag{78}$$

[15] $\tilde{m}$ is the total $\tilde{J}^3$ eigenvalue of the state so it can be lowered by applying J_{-n}^- to the highest weight state. So we have the constraint $q \geq -\tilde{N}$.

Since both N_w and q are integers, the energy spectrum is discrete. This is reasonable since we are considering the string trapped inside of AdS_3. The constraint $1/2 < \tilde{j} < (k-1)/2$ translates into the inequality

$$\frac{k}{4}w^2 + \frac{w}{2} < N_w + h - 1 + \frac{1}{4(k-2)} < \frac{k}{4}(w+1)^2 - \frac{w+1}{2}. \tag{79}$$

This is the quantum version of the semi-classical formula (59). In fact, if we take $k, h \gg \tilde{N}, q, w$, (79) reduces to (59). As in the semi-classical discussion, w is not necessarily related to the physical winding number of the string. It is just an integer labeling the type of representation that the string state is in.

The analysis for the representations coming from the continuous representations for the zero modes is similar. If we do not spectral flow, the only state in the continuous representation is the tachyon. If we do spectral flow, we get the equation (74), which can be conveniently rewritten as

$$J_0^3 = \tilde{m} + \frac{wk}{2} = \frac{kw}{4} + \frac{1}{w}\left(\frac{\frac{1}{4}+s^2}{k-2} + \tilde{N} + h - 1\right) \tag{80}$$

For continuous representations w is labeling the physical winding of the string when it approaches the boundary of AdS. In this case we do not get an equation like (76) since, for continuous representations, $\tilde{m}$ is not related to j. Comparing with the classical formula (40), we identify s as the momentum α/k of the long string along the radial direction of AdS_3. We clearly see that the energy of this state is above the threshold to produce an excitation that will approach the boundary as a w-times wound string.

We can see that, whenever the value of h is such that it saturates the range (79), we have a continuous representation with the same energy. This is clear for the lower bound in the case of $w = 0$ since, for each state in the discrete representation with $j = 1/2$, there is one in the continuous representation with the same values of L_0 and J_0^3. By the spectral flow, we see that the same is true for the lower bound in (79) for any w. Indeed we can check explicitly that a state in the discrete representation with parameters $(h, w, q, \tilde{N})$ saturating the lower bound in (79) has the same spacetime energy as a state in the continuous representation with parameters $(h, w, s = 0, \tilde{N})$. (The parameter α in the continuous representation is fixed by the value of J_0^3 in (80).) Similarly, if we have a state in a discrete representation saturating the upper bound in (79), it has the same spacetime energy as a state in the continuous representation with parameters $(h, w+1, s = 0, \tilde{N}' = \tilde{N} + q)$. Note that, since $q \geq -\tilde{N}$ (see the footnote in the previous page), we have $\tilde{N}' \geq 0$. In this case, to show that the two states have the same energy, it is useful to identify the state in $\mathcal{D}^{+,w}_{\tilde{j}=\tilde{j}}$ as a state in $\mathcal{D}^{-,w+1}_{\hat{j}=k/2-\tilde{j}}$. Since $\tilde{j} \to (k-1)/2$ corresponds to $\hat{j} \to 1/2$ under this identification, we can apply the above argument for the lower bound to show that we will find a state

in the continuous representation. The shift $\tilde{N}' = \tilde{N} + q$ comes from the fact that the identification $\mathcal{D}^{+,w}_{\tilde{j}=\tilde{j}} = \mathcal{D}^{-,w+1}_{\tilde{j}=k/2-\tilde{j}}$ involves spectral flow one more time.

The above paragraph explains what happens as we change $\tilde{j}$ in a discrete representation and we make it equal to the upper or lower bound: a continuous representation appears. Another question that one could ask is the following. Given a value of h, what is the state with the lowest value of J^3_0 that satisfies the physical state conditions? Let us first look for the lowest energy state in the discrete representations obeying the bound (79). Within this bound, one can show that $\partial J^3_0(h, w, q, \tilde{N})/\partial q \geq 0$ and $\partial J^3_0(h, w, q = -\tilde{N}, \tilde{N})/\partial \tilde{N} \geq 0$. Therefore, if we can set $q = \tilde{N} = 0$, it will give the lowest energy state in the discrete representations. This is possible if h is within the range,

$$\frac{k}{4}w^2 + \frac{w}{2} < h - 1 + \frac{1}{4(k-2)} < \frac{k}{4}(w+1)^2 - \frac{w+1}{2}. \tag{81}$$

With some more work, one can show that, for h in this range, there isn't any state in a continuous representation whose energy is lower than that of the discrete representation state with $\tilde{N} = q = 0$. As we saw in the above paragraph, at the upper or lower bound of (81), the energy of the discrete state $(q = 0, \tilde{N} = 0)$ coincides with that of the continuous state with $(s = 0, \tilde{N} = 0)$. Outside this range (81), it is not possible to set $\tilde{N} = q = 0$, and the lowest energy state will be in a continuous representation. In our semi-classical discussion in the last section, we found that the discrete representation can decay into the continuous representation at $h = kw^2/4$. Now we see that, in the fully quantum description, the range over which a continuous representation has lower energy has expanded from the point $h = kw^2/4$ to a strip of width w:

$$\frac{k}{4}w^2 - \frac{w}{2} < h - 1 + \frac{1}{4(k-2)} < \frac{k}{4}w^2 + \frac{w}{2}. \tag{82}$$

So far we have restricted our attention to right-moving sectors of the Hilbert space. Let us now discuss how the left and right movers are combined together. For the classical solution of the long string, the worldsheet periodicity requires that the spectral flow has to be done simultaneously on both the left and right movers with the same amount. If AdS_3 were not the universal cover of $SL(2, R)$ but its single cover, different amounts of the left and the right spectral flows would have been allowed since the resulting solution is periodic modulo the closed timelike curve of $SL(2, R)$. It is straightforward to identify the corresponding constraint in the quantum theory. Suppose we perform the spectral flows by the amount w_L and w_R on the left and the right-movers. A state with conformal weights (h_L, h_R) and the J^3_0 charge $(\tilde{m}_L, \tilde{m}_R)$ is mapped by this transformation to a state with conformal weights $(h_L - w_L\tilde{m}_L - \frac{k}{4}w_L^2, h_R - w_R\tilde{m}_W - \frac{k}{4}w_R^2)$, according to (69). The worldsheet locality, which is the quantum counterpart of the periodicity of the classical solution, requires that the conformal weights h_L and h_R

differ only by an integer. If this is the case before spectral flow, the same requirement after the flow implies

$$w_L \tilde{m}_L + \frac{k}{4} w_L^2 = w_R \tilde{m}_R + \frac{k}{4} w_R^2 \quad \text{(mod integer)}. \tag{83}$$

For generic values of $(\tilde{m}_L, \tilde{m}_R)$, the only solution to this constraint is $w_L = w_R$. In this paper, we are considering only the universal cover of $SL(2,R)$ as the target space of the model. In this case, the spectrum of $(\tilde{m}_L, \tilde{m}_R)$ is continuous, and only the left-right symmetric spectral flow $w_L = w_R$ is allowed.

Summary:

We propose that the spectrum of the $SL(2,R)$ WZW model (for the universal cover of $SL(2,R)$) contains the following two types of representations. First the spectral flow of the continous representations, with the same amount of spectral flow on the left and right, $\hat{\mathcal{C}}^{\alpha,w}_{1/2+is,L} \times \hat{\mathcal{C}}^{\alpha,w}_{1/2+is,R}$. Then the discrete representations $\hat{\mathcal{D}}^{+,w}_{\tilde{j},L} \times \hat{\mathcal{D}}^{+,w}_{\tilde{j},R}$ with the same amount of spectral flow on the left and right and the same value of $\tilde{j}$, with $1/2 < \tilde{j} < (k-1)/2$. In the string theory, these representations should be tensored with the states of the internal CFT, and the Virasoro constraints should be imposed.

5 Scattering of long string

When a long string comes in from the boundary of AdS_3 to the center, it will scatter back to the boundary. In this process the winding number could in principle change. In order to study the S-matrix between incoming and outgoing long strings, it is convenient to perform the rotations to Euclidean signature spaces, both on the worldsheet and in spacetime. Following the standard procedure, we define the hermiticity as is natural in the Lorentzian theory. For this reason we still have the $SL(2,R)_L \times SL(2,R)_R$ currents in the Euclidean theory. The relevant conformal field theory, whose target space is the 3-dimensional hyperbolic space $H_3 = SL(2,C)/SU(2)$ has been studied in [18, 26, 27, 28, 29, 25, 30].

5.1 Vertex operators

To compute the scattering amplitudes, we would like to find vertex operators for all representations considered above. Spectral flow is realized in the vertex operator formalism in the following standard fashion [31]. We bosonize the J^3 currents, introducing

left and right moving chiral bosons[16] through

$$J_R^3 = -i\sqrt{\frac{k}{2}}\partial\phi(z) \qquad J_L^3 = -i\sqrt{\frac{k}{2}}\bar{\partial}\phi(\bar{z}) \tag{84}$$

A state with charge m under J_R^3 contains an exponential in $\phi(z)$ of the form $e^{im\sqrt{\frac{2}{k}}\phi(z)}$. The other two currents therefore can be expressed as

$$J_R^+ = \psi e^{i\sqrt{\frac{2}{k}}\phi(z)}, \quad J_R^- = \psi^\dagger e^{-i\sqrt{\frac{2}{k}}\phi(z)}, \tag{85}$$

and similarly for $J_L^\pm$. A primary field $\Phi_{jm\bar{m}}(z,\bar{z})$ of the current algebra can be expressed as

$$\Phi_{jm\bar{m}} = e^{im\sqrt{\frac{2}{k}}\phi(z)+i\bar{m}\sqrt{\frac{2}{k}}\phi(\bar{z})}\Psi_{jm\bar{m}}, \tag{86}$$

where $\Psi_{jm\bar{m}}$ carries no charges with respect to $J_{R,L}^3$. In the case of the $SU(2)$ model, the field corresponding to Ψ is known as a parafermion. The parafermion for the $SL(2,R)$ model was studied in [32]. The conformal weights of the parafermion field $\Psi_{jm\bar{m}}$ is

$$\begin{aligned} h_{\Psi;jm\bar{m}} &= -\frac{j(j-1)}{k-2} + \frac{m^2}{k} \\ \bar{h}_{\Psi;jm\bar{m}} &= -\frac{j(j-1)}{k-2} + \frac{\bar{m}^2}{k}. \end{aligned} \tag{87}$$

In the discrete lowest weight representation, $m, \bar{m} = j, j+1, j+2, \cdots$. In particular, when $j = k/2$, the field $\Psi_{j=k/2,m=\bar{m}=k/2}$ has conformal weights $h = \bar{h} = 0$. Since the parafermion field lives in the unitary conformal field theory it is natural to assume that it is the identity operator[17]. Here we simply note that the operator

$$e^{i\sqrt{\frac{k}{2}}(\phi(z)+\phi(\bar{z}))}$$

has the correct OPE for the primary field of spin $j = k/2$ with the $SL(2,R)$ currents.

Using the parafermion notation, the operator obtained by the spectral flow by w units is expressed as

$$\Phi^w = e^{i(\tilde{m}+wk/2)\sqrt{\frac{2}{k}}\phi(z)+i(\tilde{\bar{m}}+wk/2)\sqrt{\frac{2}{k}}\phi(\bar{z})}\Psi_{j\tilde{m}\tilde{\bar{m}}} \tag{88}$$

It is easy to see that the conformal weight is given by

$$L_0 = \frac{-j(j-1)}{k-2} - mw + kw^2/2 \tag{89}$$

[16]Reflecting the hermiticity of the $SL(2,R)$ model, the scalar field ϕ is hermitian, but with a wrong sign for the two-point function $\langle\phi(z)\phi(z')\rangle = \log(z-z')$.

[17]Recently we have learned that a similar argument has appeared in unpublished notes by A. B. Zamolodchikov. We thank him for having his note available to us [33].

5.2 Reflection coefficient

We will compute the amplitude, using the formulae obtained in [34, 35, 26, 33], in the case that the winding number does not change.

The long string states are in the spectral flow of the continuum representation. The corresponding vertex operators are

$$\begin{aligned} \Phi^j_{m\bar{m}} =& e^{m\phi(z)+\bar{m}\phi(\bar{z})}\Psi^j_{\tilde{m}\tilde{\bar{m}}}V_{h\bar{h}}(z,\bar{z}) \ , \\ \tilde{m} = m - wk/2 \ , \qquad \tilde{\bar{m}} = \bar{m} - wk/2 \ , \qquad & j = \frac{1}{2} + is \end{aligned} \tag{90}$$

where $V_{h\bar{h}}$ is an operator in the internal part with conformal weights $(h,\bar{h})$. The physical energy E and angular momentum ℓ of a state in AdS_3 are given by

$$m = \frac{1}{2}(E+\ell), \qquad \bar{m} = \frac{1}{2}(E-\ell). \tag{91}$$

The physical state constraint is (80) with $\tilde{N} = 0$. This implies that

$$\tilde{m} = -wk/4 + \frac{1}{w}\left[\frac{1/4+s^2}{k-2} + h - 1\right] \tag{92}$$

Now we can now consider the two point function [26, 27, 33]

$$\begin{aligned} \langle \Phi^j_{m\bar{m}}(z,\bar{z})\Phi^{j'}_{m'\bar{m}'}(z',\bar{z}')\rangle =& \frac{\Gamma(1/2+is-\tilde{m})\Gamma(1/2+is+\tilde{\bar{m}})\Gamma(-2is)\Gamma(\frac{2is}{k-2})}{\Gamma(1/2-is-\tilde{m})\Gamma(1/2-is+\tilde{\bar{m}})\Gamma(2is)\Gamma(-\frac{2is}{k-2})} \\ & \times \delta(s-s')\delta_{N+N'}\delta(E+E') \end{aligned} \tag{93}$$

The z dependence is just $1/|z-z'|^4$ coming from the fact that the two operators have weight (1,1). This is the reflection amplitude and the values of $\tilde{m}, \tilde{\bar{m}}$ are determined by (92)(notice that m is the physical energy, not $\tilde{m}$.).

As explained in [28] in this context, in string theory we have to integrate over z and divide by the volume of $SL(2,C)$. We can use $SL(2,C)$ invariance to put $z=0$, $z'=\infty$ in the correlator. The volume of the rest of $SL(2,C)$ then gives $\int \frac{d^2z}{|z|^2}$, which cancels one of the delta-functions in (93). Notice that $\delta(s-s')\delta(E+E') = \delta(s-s')\delta(0)$, the volume of $SL(2,C)$ cancels the $\delta(0)$ piece.

Now if we study the poles of (93), we find that they are located at $1/2+is-\tilde{m} = -q$ with $q = 0,1,2,\cdots$. They come from the first Gamma-function. Taking this condition together with (92) we find that

$$1/2 + is + q = \tilde{m} = -wk/4 + \frac{1}{w}\left[\frac{1/4+s^2}{k-2} + h - 1\right] \tag{94}$$

and this equation is precisely the same as the usual mass shell equation for discrete states if we take $\tilde{j} = 1/2 + is$. There are similar poles from the second Gamma-function. There are no poles coming from the third factor since they cancel extra poles appearing in the other factors. Notice that the poles appearing in (94) satisfy precisely the equation (76) for bound states in the representation $\widehat{\mathcal{D}}^{+,w}_{\tilde{j}}$ (with $\tilde{N} = 0$). There is however an important difference. In (76) the value of $\tilde{j}$ obeyed the condition

$$\frac{1}{2} < \tilde{j} < \frac{k-1}{2} \tag{95}$$

while we do not have such a condition in (94). It is interesting to note that if $\tilde{j}$ satisfies (95) then the residue at the pole has the proper sign to be interpreted as coming from a bound state. When $\tilde{j} = (k-1)/2$, *i.e* at the upper bound of (95), we find that there is no pole. Moreover, immediately above that value, we have the wrong sign for the pole residue. This might make us worry that the amplitude is not having the right analytic structure. However, in order to have a one-to-one correspondence between poles of the scattering amplitude and bound states, the potential has to decrease sufficiently rapidly at the infinity [36], a condition that is not met in our case. In such a situation, it is possible to have extra poles that do not correspond to physical states. We plan to analyze the poles and their implications for physical states in a future publication.

5.3 Relation to the scattering off the two-dimensional black hole

The coset of the $SL(2,R)$ WZW by the $U(1)$ generated by J^3 gives a sigma-model whose target space is the two-dimensional black hole with the Euclidean signature metric [37]. The geometry of the black hole is like a semi-infinite cigar with an asymptotic region in the form of the cylinder $\mathbb{R} \times S^1$. The dilaton field grows as one approaches the center of the black hole, but it remains finite since the geometry is terminated at the tip of the cigar. The string theory on $SL(2,R)/U(1)\times(\text{time})\times\mathcal{M}$ is closely related to the string theory on $AdS_3 \times \mathcal{M}$ since the physical state conditions for the latter implies $J^3_n|\text{physical}\rangle = 0$ for $n \geq 1$, as we show in Appendix A. Similarly the superstring theory on $AdS_3 \times \mathcal{M}$ is related to the Kazama-Suzuki coset $SL(2,R)/U(1)$.

There is however difference between the zero mode sectors of the theories on AdS_3 and on the two-dimensional black hole. In order to construct representations for $SL(2)/U(1)$, we can start from the representations of $\widehat{SL}(2,R)$ that we described above and impose the condition that $J^3_{n>0}$ annihilate the state and that the total AdS_3 energy vanishes, $J^3_{0,R} + J^3_{0,L} = m + \bar{m} = 0$. In terms of the parafermion $\Psi_{j\tilde{m}\tilde{\bar{m}}}$ given in (86) and (88), the condition is $\tilde{m} + \tilde{\bar{m}} = wk$. The locality condition $m - \bar{m} = n$ where n is an integer implies that $\tilde{m} - \tilde{\bar{m}} = n$. These two quantization conditions are the ones in [38]

(see equation 3.6 of that paper). The $SL(2)/U(1)$ theory has been studied recently in connection with "little" string theories in [24, 39].

6 Conclusion

In this paper, we studied the physical spectrum of bosonic string theory in AdS_3. We proposed that the complete Hilbert space of the $SL(2,R)$ WZW model consists of the continuous representations and their spectral flow $\widehat{\mathcal{C}}^{\alpha,w}_{j=1/2+is} \times \widehat{\mathcal{C}}^{\alpha,w}_{j=1/2+is}$, and the discrete representations and their spectral flow $\widehat{\mathcal{D}}^{\pm,w}_j \times \widehat{\mathcal{D}}^{\pm,w}_j$ with the constraint $1/2 < j < (k-1)/2$. The sum over the spectral flow is required if we assume that the Hilbert space realizes the full loop group of $SL(2,R)$, including its topologically non-trivial elements. We found that this proposal leads to the physical spectrum of the string theory with the correct semi-classical limits.

In particular, we have solved the two puzzles which we mentioned in the introduction. The no-ghost theorem for $\widehat{\mathcal{D}}^{\pm}_j$ requires the constraint $0 < j < k/2$. If we only had the unflowed sector (with $w = 0$), it would imply the upper bound on allowed mass of string states, which appears artificial. This was one of the puzzles. We have resolved this puzzle by showing that the upper bound on the mass is removed if we include all the spectral flowed sectors in the Hilbert space. Moreover we showed that the consistency with the spectral flow and the standard harmonic analysis of the zero modes requires the constraint $1/2 < j < (k-1)/2$, more stringent than the one required by the no-ghost theorem. The constraint $1/2 < j < (k-1)/2$ is found to be consistent with the locations of the poles in the reflection coefficient (with the correct sign for the pole residues; see also [24]) and the modular invariance of the partition function.

Another puzzle was to identify states in the Hilbert space corresponding to the long strings. We found that these states are in the spectral flow of the continuous representations, $\widehat{\mathcal{C}}^{\alpha,w}_{j=1/2+is} \times \widehat{\mathcal{C}}^{\alpha,w}_{j=1/2+is}$. The integer w, which parametrized the amount of the spectral flow, is identified with the winding number of the long string stretched closed the boundary of AdS_3. The physical spectrum of the long strings obtained from these representations agrees with the expectations from the semi-classical analysis in [11, 12].

The resolutions of these puzzles removes the longstanding doubts about the consistency of the model. Moreover it appears that the $SL(2,R)$ WZW model is exactly solvable, just as WZW models for compact groups, although its Hilbert space structure is significantly different from those of the compact cases. We hope that further study of the model will provide us more useful insignts into the AdS/CFT correspondence and strings in curved spaces in general.

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A No-ghost theorems

In this appendix we would like to extend the proof of the no-ghost theorem to all the representations considered above. We assume $k > 2$. The proof of the no-ghost theorem for the standard lowest energy representations [2, 3, 4, 5, 6, 7, 8, 9, 22] involves two parts. Part I consists of showing that a physical state can be chosen, up to a null state, to be such that $J^3_n|\psi >= 0$, for $n \geq 1$. This first part uses $0 < j < k/2$ for the $\mathcal{D}^{\pm}_j$ representations as well as $c = 26$ and the mass shell condition. This was shown in [2, 3, 4, 5, 6, 7, 8, 9]. Part II consists in showing that any state that is annihilated by $J^3_{n>0}$ has non negative norm. This step also uses $0 < j < k/2$ for the $\mathcal{D}^{\pm}_j$ representations. This was done in [22]. Here we will use the same strategy and prove Part I for the all our representations. The no-ghost theorem then follows from Part II.

We first review the proof of Part I for the representations with $w = 0$ and then we do Part I for the $w \neq 0$ representations.

A.1 Proof of Part I for unflowed representations

Here we follow the proof in [2, 3, 6, 7, 9]. It has essentially three steps.

Step 1: The first step of the proof is to show that states of the form

$$\begin{aligned} &L_{-n_1}L_{-n_2}\cdots L_{-n_N}J^3_{-m_1}J^3_{-m_2}\cdots J^3_{-m_M}|f\rangle \\ &\quad n_1 \geq n_2 \geq \cdots \geq n_N, \quad m_1 \geq m_2 \geq \cdots \geq m_M, \\ &\text{with } L_n|f\rangle = J^3_n|f\rangle = 0 \quad \text{for} \;\; n \geq 1, \end{aligned} \tag{96}$$

are linearly independent and that they form a complete basis of the Hilbert space.

The states $|f\rangle$ are constructed from states in the current algebra times some states in an internal conformal field theory. This internal piece is assumed to be unitary. This step involved separating the piece of L_n involving $L^{(3)} =: J^3J^3 :$, defining $\hat{L}_n = L_n - L^{(3)}_n$. One can show that the states (96) are in one to one correspondence with states of the form

$$\begin{aligned} &L_{-n_1}L_{-n_2}\cdots L_{-n_N}J^3_{-m_1}J^3_{-m_2}\cdots J^3_{-m_M}|f\rangle \\ &\quad n_1 \geq n_2 \geq \cdots \geq n_N, \quad m_1 \geq m_2 \geq \cdots \geq m_M, \end{aligned} \tag{97}$$

Notice that the conditions (96) on $|f\rangle$ are the same as $\hat{L}_{n>0}|f\rangle = J^3_{n>0}|f\rangle = 0$. It is easier to show that (97) is a basis since now we can think of the CFT as a product of a $U(1)$ factor with the rest. The rest is a CFT with $c = 25$ and therefore the fact that (97) is a basis reduces to showing that there are no null states in the Virasoro descendents on a primary field. This will be true if the conformal weight of the rest is

positive. This reduces to showing that $c_2/(k-2) + m^2/k + M > 0$, where M is the grade in the $SL(2,R)$ piece. For the continuous representations, this is obvious since $c_2 > 0$. For lowest weight representations, this inequality can be shown by rewriting it as

$$\frac{2j\,(k/2-j)}{k(k-2)} + \frac{2M}{k}\left(\frac{k}{2}-j\right) + \frac{2j}{k}(-j+m+M) + \frac{1}{k}(j-m)^2 > 0 \tag{98}$$

We to use $0 < j < k/2$ and also the fact that $m \geq j - M$, which is true in general. Notice that the m that appears here is the total J_0^3 value, after we applied $J_{-n}^{\pm}$ any number of times. Notice that in this step we did not use that the states were obeying the mass shell condition, but we used $0 < j < k/2$ and that $c = 26$.

Step 2: Here we show that a physical state can be chosen so that it involves no L_{-n} when written as (96).

A physical state can be written as a state with no L_{-n} plus a spurious state. A spurious state is a state with at least one L_{-n}. Then we use the fact that, when $c = 26$, L_n $(n \geq 1)$ acting on a spurious state which satisfies the $L_0 = 1$ condition leaves it as a spurious state [40, 41] . If $L_{n>0}$ acts on a state of the form (96) with no L_{-n} then it will not produce any L_{-n}. Together with the fact that (96) is a basis this implies that the part of the state with no L_{-n} satisfies the physical state condition on its own, and therefore the rest is a null state (a spurious physical state).

Step 3: We show that if the physical state $|\psi\rangle$ involves no L_{-n} when written as in (96) then $J_n^3|\psi\rangle = 0$.

Since there are no L_{-n}'s in the physical state ψ this implies that $L_n^{(3)}\psi = 0$ for $n \geq 1$. Then we try to show that the only states satisfying this will be states with $J_n^3\psi = 0$ for $n \geq 1$. This would be true if there are no null states in the $L^{(3)}$ Virasoro descendents of the states $|f\rangle$ we considered above. If $m \neq 0$ then one can show that there is no null state in the Virasoro descendents in the $L^{(3)}$ Virasoro descendents. There are two states with $m = 0$ one is in the continuous representation, but the mass shell condition automatically implies that $N = 0$ (there are no J_{-n}^a in this state) and therefore the state has positive norm. The other is the state in the lowest weight representation

$$J_{-1}^{-}|j=1\rangle \tag{99}$$

(and of course its complex conjugate in the highest weight representation). This state has positive norm. Note that m is the physical energy in AdS_3 of the state in question. Zero energy states, therefore imply that we have a normalizable zero mode. This is the state corresponding to the identity operator in the spacetime boundary conformal field theory, the state $\bar{J}J\Phi_1$ of [25] which played an important role in the computation of the spacetime Virasoro algebra.

One can show, using the mass shell condition, that all other states have $m \neq 0$. The mass shell condition is

$$-\frac{j(j-1)}{k-2} + N + h' - 1 = 0 \tag{100}$$

where N is the grade in the $SL(2,R)$ part and h' is the conformal weight of the rest, $h' \geq 0$. If $0 < j < 1$ then m is nonzero because it can only change by an integer by the action of the $J_n^{\pm}$ currents. If $j = 1$ with $N = 1$ and $h' = 0$ we find (99)and states with positive m.

Consider now $j > 1$. If we had $m = 0$ then we also need $N \geq j$, $j \geq 2$ (since $m = 0$ only if j is integer) and furthermore

$$-\frac{j(j-1)}{k-2} + N - 1 \geq \frac{(j-1)(k-2-j)}{k-2} > 0 \tag{101}$$

provided $j \leq k/2$. Since j has to be at least 2, then $k > 4$ and therefore $k-2-k/2 > 0$. Thus we conclude that (100) would not be obeyed if $m = 0$.

A.2 Proof of Part I for flowed representations

Now we would like to generalize the above discussion to the spectral flowed representations that we called $\hat{\mathcal{C}}^{\alpha,w}_{1/2+is}$ and $\hat{\mathcal{D}}^{+,w}_{\tilde{j}}$. In the case of discrete representations we want to show that the no ghost theorem holds for $0 < \tilde{j} < k/2$ where $\tilde{j}$ labels the representation before we perform the spectral flow operation, i.e. it labels a representation of the current algebra with $\tilde{L}_0$ bounded below. So we consider the same representations we had above but we modify the physical state conditions. This is equivalent to imposing the usual conditions on the flowed representations. We would like to prove that, given any state built on a lowest weight or continuous representation with respect to $\tilde{J}_n$, the physical state condition $(L_n - \delta_{n,0})|\psi\rangle = 0$ $n \geq 0$ with respect to L_n removes non-negative norm states. We only consider spectral flow with $w > 1$ on continuous or lowest weight representations $\hat{\mathcal{D}}^{+}_{\tilde{j}}$. These and their complex conjugates cover all the representations we needed to consider. We reproduce now the steps in A.1.

Step 1: In (96) we need to show that they form a basis with $L_{-n} = \tilde{L}_{-n} - w\tilde{J}^3_{-n}$. We know that they would form a basis if we had an expression like (96)with $L_{-n} \to \tilde{L}_{-n}$. Fortunately there is an invertible one to one map between these two sets of states, so that they form a basis.

Step 2: It is the same since only $c = 26$ is used.

Step 3: If we write a physical state, $|\psi\rangle$, as a state with no L_{-n} then $L^{(3)}_n$ with $n \geq 1$ annihilates it. Again we will try to show that $m = \tilde{m} + kw/2$ is nonzero and that will

imply that $J^3_{n>0}|\psi\rangle = 0$. For this we need to use the new mass shell condition

$$\frac{\tilde{c}_2}{k-2} + \tilde{N} + h' - wm + \frac{kw^2}{4} = 1 \tag{102}$$

where $\tilde{N}$ is the level inside the current algebra before the spectral flow, $\tilde{c}_2$ is the second casimir in terms of $\tilde{j}$ and h' is the conformal weight of state in the internal conformal theory (the internal piece needs not be a primary state, and we only require that the whole combined state needs to be primary). We can assume with no loss of generality that $w \geq 1$. Let us start with the spectral flow of a continuous representation, (102)implies that if $m = 0$ then $\tilde{N} = 0$ and and there are no negative norm states. (The only solution with $m = 0$ is in the case of $k = 3$ and $\tilde{j} = 1/2$).

Let us turn to lowest weight representations. Thanks to the restriction $0 < \tilde{j} < k/2$, we have $\tilde{c}_2/(k-2) > -k/4$. Therefore, if $m = 0$, the left hand side of (102) is larger than $k/4(w^2 - 1)$. If $w \geq 2$, (102) cannot be obeyed. If $w = 1$, $m = 0$ implies $\tilde{m} = -k/2$ and $\tilde{N}$ in (102) has to be at least $\tilde{N} \geq \tilde{j} + k/2$. However, in this case we find $\tilde{c}_2/(k-2) + \tilde{N} + k/4 \geq k/2 + \tilde{j} > 1$ (here we used $k > 2$) and again (102) is not satisfied.

So we conclude that all states can be mapped into states obeying $J^3_{n>0}|\psi\rangle = 0$.

B Partition function

In this Appendix, we discuss the partition function of the $SL(2,R)$ WZW model and its modular invariance.

B.1 Partition function of the SU(2) model

Before we begin discussing the modular invariance of the $SL(2,R)$ theory, let us review the case of $SU(2)$.

The characters $\chi^k_l(\tau,\theta)$ $(l = 0, \frac{1}{2}, 1, \cdots \frac{k}{2})$ of the irreducible representations of the $SU(2)_k$ affine algebra transform under the modular transformation as

$$\chi^k_l(-1/\tau, -\theta/\tau) = \exp\left(2\pi i \frac{k}{4}\frac{\theta^2}{\tau}\right) \sum_{l'} S_{ll'} \chi^k_{l'}(\tau,\theta), \tag{103}$$

where $S_{ll'}$ is some orthonormal $(k+1)\times(k+1)$ matrix. The diagonal (so-called A_k-type) modular invariant combination is therefore

$$e^{-2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}} \sum_l |\chi_l(\tau,\theta)|^2. \tag{104}$$

The exponential factor $e^{-2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}}$ is there to cancel the exponential factor in (103) as

$$\frac{[Im(-\theta/\tau)]^2}{Im(-1/\tau)} = \frac{(Im\theta)^2}{Im\tau} + i\frac{\theta^2}{2\tau} - i\frac{\bar{\theta}^2}{2\bar{\tau}}. \tag{105}$$

It is known that the exponential factor in (104) is a consequence of the chiral anomaly and therefore of the OPE singularity,

$$J^3(z)J^3(w) \sim \frac{k/2}{(z-w)^2}. \tag{106}$$

B.2 Partition function of the SL(2,C)/SU(2) model

In string theory, one-loop computations are done after performing the Euclidean rotation on both the target space and the worldsheet (or stay in the Lorentzian signature space and use the $i\epsilon$ prescription). The modular invariance of the partition function is imposed on the Euclidean worldsheet. In our case, the Euclidean rotation of the target space means $SL(2,R) \to H_3 = SL(2,C)/SU(2)$. The partition function of the $SL(2,C)/SU(2)$ model has been evaluated in [18] as

$$Z_{SL(2,C)/SU(2)} \sim \frac{1}{\sqrt{Im\tau}e^{\frac{-2\pi(Im\theta)^2}{Im\tau}}|\vartheta_1(\tau,\theta)|^2}. \tag{107}$$

Note that our definition of the partition function differs from that in [18] by the factor $e^{2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}}$. It apears that, without this factor, the partition function is not modular invariant[18]. One may expect that this partition function is related to the one for the $SL(2,R)$ model by the Euclidean rotation. In the discussion below, we first evaluate the $SL(2,R)$ partition function on the Lorentzian torus, and therefore take $\tau, \bar{\tau}, \theta, \bar{\theta}$ to be independent real variables. We then analytically continue them to complex values so that (τ, θ) are complex conjugate of $(\bar{\tau}, \bar{\theta})$. We will find that, by doing this analytic continuation, and ignoring contact terms, the $SL(2,R)$ partition function turns into the $SL(2,C)/SU(2)$ partition function (107), provided we impose the constraint $1/2 < j < (k-1)/2$ on the discrete representations.

[18]The puzzle about the apparent lack of the the modular invariance was recently resolved in [42].

B.3 Discrete representations of SL(2,R)

The character of the discrete representation D_j^+ is

$$\begin{aligned}\chi_j^+(\tau,\theta) =& Tr(e^{2\pi i\tau(L_0-\frac{k}{8(k-2)})}e^{2\pi i\theta J_0^3})\\ =&\frac{\exp\left[2\pi i\tau\left(-\frac{j(j-1)}{k-2}-\frac{k}{8(k-2)}\right)+2\pi i\theta j\right]}{(1-e^{2\pi i\theta})\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})(1-e^{2\pi in\tau}e^{2\pi i\theta})(1-e^{2\pi in\tau}e^{-2\pi i\theta})}\\ =&\frac{\exp\left[-\frac{2\pi i\tau}{k-2}(j-\frac{1}{2})^2+2\pi i\theta(j-\frac{1}{2})\right]}{i\vartheta_1(\tau,\theta)}.\end{aligned} \tag{108}$$

where $\vartheta_1(\tau,\theta)$ is the elliptic theta-function

$$\vartheta_1(\tau,\theta) = -i\sum_{n=-\infty}^{\infty}(-1)^n\exp\left[\pi i\tau\left(n-\frac{1}{2}\right)^2+2\pi i\theta\left(n-\frac{1}{2}\right)\right]. \tag{109}$$

The spectral flow

$$\tilde{L}_0 = L_0 + wJ_0^3 - \frac{k}{4}w^2, \quad \tilde{J}_0^3 = J_0^3 - \frac{k}{2}w, \quad (w = 0, \pm1, \pm2, \cdots), \tag{110}$$

transforms the character χ_j^+ as

$$\begin{aligned}&Tr(e^{2\pi i\tau(\tilde{L}_0-\frac{k}{8(k-2)})}e^{2\pi i\theta\tilde{J}_0^3})\\ =&Tr(e^{2\pi i\tau(L_0+wJ_0^3-\frac{k}{4}w^2-\frac{k}{8(k-2)})}e^{2\pi i\theta(J_0^3-\frac{k}{2}w)})\\ =&\frac{\exp\left[-2\pi i\tau\left(\frac{(j-\frac{1}{2})^2}{k-2}-w(j-\frac{1}{2})+\frac{k}{4}w^2\right)+2\pi i\theta(j-\frac{1}{2}-\frac{k}{2}w)\right]}{i\vartheta_1(\tau,\theta+w\tau)}\\ =&(-1)^w\frac{\exp\left[-\frac{2\pi i\tau}{k-2}(j-\frac{1}{2}-\frac{k-2}{2}w)^2+2\pi i\theta(j-\frac{1}{2}-\frac{k-2}{2}w)\right]}{i\vartheta_1(\tau,\theta)},\end{aligned} \tag{111}$$

where we used

$$\vartheta_1(\tau,\theta+w\tau) = (-1)^w\exp\left(-\pi i\tau w^2-2\pi i\theta w\right)\vartheta_1(\tau,\theta). \tag{112}$$

We have also performed an analytic continuation such as

$$\sum_{n=0}^{\infty}q^n = -\sum_{n=1}^{\infty}q^{-n},$$

ignoring terms like $\sum_{n=-\infty}^{\infty}q^n \sim \delta(\tau)$. From here on, we allow (τ,θ) to take complex values and $(\bar{\tau},\bar{\theta})$ to be their complex conjugates.

Let us sum over allowed representation. According to our proposal about the Hilbert space of the WZW model, all the representations in the allowed range $1/2 < j <$

$(k-1)/2$ should appear. We also require that the spectrum to be invariant under the spectral flow (110), so we need to sum over w. The part of the partition function made by discrete representations is then

$$
\begin{aligned}
& e^{+2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}} \sum_{w=-\infty}^{\infty} \int_{1/2}^{(k-1)/2} dj \frac{\exp\left[\frac{4\pi Im\tau}{k-2}(j-\frac{1}{2}-\frac{k-2}{2}w)^2 - 4\pi Im\theta(j-\frac{1}{2}-\frac{k-2}{2}w)\right]}{|\vartheta_1(\tau,\theta)|^2} \\
& = e^{+2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}} \int_{-\infty}^{\infty} dt \frac{\exp\left[\frac{4\pi Im\tau}{k-2}t^2 - 4\pi Im\theta t\right]}{|\vartheta_1(\tau,\theta)|^2} \\
& \sim \frac{1}{\sqrt{Im\tau}e^{-2\pi\frac{(Im\theta)^2}{Im\tau}}|\vartheta_1(\tau,\theta)|^2}.
\end{aligned}
\tag{113}
$$

It is interesting to note that the j-integral over the range $1/2 < j < (k-1)/2$ and the sum over w fit together to give the t-integral over $-\infty < t < \infty$. Since the spectral flow with $w = 1$ maps D_j^+ to $D_{k/2-j}^-$, we do not have to consider the orbit of D_j^- separately. The exponential factor $e^{+2\pi\frac{k}{2}\frac{(Im\theta)^2}{Im\tau}}$ is due to the chiral anomaly, as in the $SU(2)$ case. The sign in the exponent is opposite here since the sign of the OPE of J^3 is opposite in the $SL(2,R)$ case.

The partition function computed in (113) is manifestly modular invariant. In fact, it is identical to (107) computed for the $SL(2,C)/SU(2)$ model. This gives an additional support for our claim that the Hilbert space of the $SL(2,R)$ model contains the discrete representations of $1/2 < j < (k-1)/2$ and their spectral flow.

The construction of the partition function here is closely related to the one given in [13]. There, instead of the integral over j in (113), the partition function was given by a sum over integral values of j. This is because they considered the string theory on the single cover of the $SL(2,R)$ group manifold with the closed timelike curve. The resulting partition function, after analytic continuation, is also modular invariant and appears to be a correct one for such a model. It is, however, different from the partition function (107) of the $SL(2,C)/SU(2)$ model, as it should since the Euclidean rotation of the $SL(2,C)/SU(2)$ model is naturally related to the model on the universal cover of $SL(2,R)$ rather than on its single cover.

B.4 Continuous representations

It is curious that the sum over the discrete representations and their spectral flow alone reproduces the partition function of the $SL(2,C)/SU(2)$ model. In fact, the sum over the continuous representations and their spectral flow, although formally modular invariant by itself, does not contribute to the partition function if we assume the analytic continuation in $\tau, \bar{\tau}, \theta, \bar{\theta}$ and ignore contact terms.

The character of the continuous representation is parametrized by a pair of real numbers (s, α) with $0 \leq \alpha < 1$ and s arbitrary. The character is given by

$$\chi_{j=1/2+is,\alpha} = \eta^{-3} e^{2\pi i \frac{s^2}{k-2}\tau} e^{i\alpha\theta} \sum_n e^{2\pi i n\theta}. \tag{114}$$

As before, we regard the worldsheet metric to be of the Minkowski signature, and θ is real. So the sum $\sum_n$ in the definition of $\chi_{j,\alpha}$ gives the periodic delta-function

$$\sum_n e^{2\pi i n\theta} = 2\pi \sum_m \delta(\theta + m). \tag{115}$$

After the spectral flow (110), the character becomes

$$\chi_{j=1/2+is,\alpha;w} = \eta^{-3} e^{2\pi i \left(\frac{s^2}{k-2} + \frac{k}{4}w^2\right)\tau} 2\pi \sum_m e^{2\pi i m(\alpha - \frac{k}{2}w)} \delta(\theta + w\tau + m). \tag{116}$$

Now let us take $|\chi_{1/2+is,\alpha;w}|^2$ and integrate over s and α. The integral over α forces $m_L = m_R$ in the summation in (116). The integral over s gives the factor $1/\sqrt{Im\tau}$. So we have

$$\int_{-\infty}^{\infty} ds \int_0^1 d\alpha |\chi_{1/2+is,\alpha;w}|^2 = e^{-4\pi Im\tau \frac{k}{4}w^2} \frac{1}{\sqrt{Im\tau}|\eta|^6} \sum_m \delta^{(2)}(\theta + w\tau + m). \tag{117}$$

Let us sum this over w. We get a non-zero result only when there is some integer w such that

$$w = -\frac{Im\theta}{Im\tau}. \tag{118}$$

Therefore

$$\begin{aligned} & e^{+2\pi \frac{k}{2} \frac{(Im\theta)^2}{Im\tau}} \sum_w \int_{-\infty}^{\infty} ds \int_0^1 d\alpha |\chi_{1/2+is,\alpha;w}|^2 \\ & = \frac{1}{\sqrt{Im\tau}|\eta|^6} \sum_{w,m} \delta^{(2)}(\theta + w\tau + m). \end{aligned} \tag{119}$$

This expression is formally modular invariant since $\sum_{w,m}$ sums over the modular orbit of the delta-function and $1/|\eta|^4$ cancels its modular weight. If we assume the analytic continuation, terms of this form are all set equal to zero. So, in this sense, the continuous representation does not contribute to the partition function of the $SL(2,C)/SU(2)$ theory after the Euclidean rotation.

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