Quantum corrections to dilute Bose liquids

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It was recently shown [A. Bulgac, Phys. Rev. Lett. **89**, 050402 (2002)] that an entirely new class of quantum liquids with widely tunable properties could be manufactured from bosons, fermions, and their mixtures by controlling their interaction properties by means of a Feshbach resonance. Since the quantum fluctuations in this regime could, in principle, destabilize these objects, we extend the previous mean-field analysis of these quantum liquids by computing the lowest-order quantum corrections to the ground-state energy and the depletion of the Bose-Einstein condensate and by estimating the higher-order corrections as well. We show that the quantum corrections are relatively small, do not lead to the destabilization of the droplets and are controlled by the diluteness parameter $\sqrt{n|a|^3} \ll 1$, even though strictly speaking in this case there is no low density expansion.

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Among the quantum fluids observed so far, only ^{3}He, ^{4}He, and atomic nuclei are self-bound/liquid systems. It is very likely that this relatively short list could become extremely long by the addition of boson droplets (boselets), fermions droplets (fermilets), and their mixtures (ferbolets) [1], which owe their existence to the possibility of tuning two-atom interactions and to some unusual properties of the three-body system.

It was predicted about ten years ago [2] and was confirmed experimentally about five years ago [3] that by immersing atoms in a magnetic field (and, in principle, in electric or laser fields as well), one can alter essentially at will the scattering length a between two atoms. Under such circumstances, it is possible to create a situation where the scattering length is negative and significantly larger than the interaction radius. In the regime where the magnitude of the scattering length is much larger than the typical scale of atomic interactions r_0, the three-atom system exhibits some very unusual and interesting phenomena [4,5]. The three-body observables depend on a new scale a_3 which is not determined by any two-body observable. In the language of effective field theory, this corresponds to the fact that a three-body force appears at leading order of the low-energy expansion [6]. Also, the three-body observables are periodic functions of $\ln(a/a_3)$ and the three-particle scattering amplitude can be made arbitrarily large by fine tuning a. By choosing a negative, large ($|a| \gg r_0$), and with a three-body bound state close to the threshold, a rather unique situation is created: Two low-energy atoms experience an effective attraction, but at the same time three atoms will experience an effective repulsion. This situation is very well understood theoretically and it is known as the Efimov effect [4,5]. A large ensemble of such atoms can condense in the same way as water droplets condense from vapor. Since such condensation can occur for any atom species for which a Feshbach tunable resonance exists, an entirely new class of (bose, fermi, or mixtures) quantum liquids [7] can be created. Their equilibrium density is determined by the interplay between the two-body attraction and the three-body repulsion between particles in the Efimov regime described above. Since the magnitudes of the two-body attraction and of the three-body repulsion are to a significant extent under experimental control, the basic properties of these new quantum liquids are widely tunable [8].

In Ref. [1], the basic properties of this new class of quantum fluids were established in the mean-field approximation. It is not at all obvious that quantum fluctuations could not destabilize a boselet. It is generally expected that the mean-field approximation is rather accurate for dilute systems [9–11]. Under normal circumstances (when $a>0$ and $a \approx r_0$), it is sufficient to take into account only the binary atom-atom collisions and the effects of quantum fluctuations, whereas the triple atom and higher collisions are rather small and controlled by the diluteness parameter $\sqrt{na^3} \ll 1$ [9–11]. The regime $a<0$ was considered until recently as intrinsically unstable towards collapse, unless the number of bosons is smaller than about 1500 or so in a trap [9,12]. In the specific Efimov regime, we are interested in the fact that the two-body scattering amplitude is large ($a<0, |a| \gg r_0$), but the three-body amplitude is even larger. Even though the system is dilute with respect to the two-body collisions ($n|a|^3 \ll 1$), it is not obvious that diluteness with respect to the three-body collisions is achieved and one might suspect that the mean-field approximation could be violated. It is imperative to understand the character and the magnitude of these corrections to the ground-state properties in the case of these new quantum liquids and determine whether quantum fluctuations can lead to instabilities. One can naturally expect that such effects would perhaps manifest themselves particularly strongly in the case of bosons and less so in the case of fermions. For this reason, we shall focus our attention here on Bose systems only.

We start with the Lagrangian density

$$\mathcal{L}+\mu\mathcal{N}=\psi^\dagger\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m}+\mu\right)\psi-\frac{g_2}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{g_3}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi, \tag{1}$$

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where $\psi^\dagger$ and ψ are the creation and annihilation operators for a boson, μ is the chemical potential, and the coupling constants are

$$g_2=\frac{4\pi\hbar^2 a}{m}, \tag{2}$$

$$g_3=\frac{12\pi\hbar^2 a^4}{m}\left[d_1+d_2\tan\left(s_0\ln\frac{a}{a_3}+\frac{\pi}{2}\right)\right]=\frac{6\pi\hbar^2 a^4}{m}\Upsilon. \tag{3}$$

Here, a is the two-body scattering length and $s_0\approx 1.006\,24$. d_1 and $d_2<0$ are universal constants, whose numerical values have been determined recently [13]. The values of the coupling constant are determined by considering the scattering of two and three particles at zero momentum. a_3 is the value of the two-body scattering length for which a three-body bound state has exactly zero energy. Unlike $d_{1,2}$ and s_0, the parameter a_3 is system dependent and is also a genuine three-body characteristic. The rest of the symbols have their usual meaning. We have introduced a new dimensionless quantity $\Upsilon\gg 1$ in Eq. (3), which will prove very convenient in power counting. We will only use the Lagrangian in Eq. (1) for particle momenta much smaller than $\hbar/|a|$ so that it is legitimate to subsume the complicated dynamics on the scale $\hbar/a$, leading to the Efimov effect into a contact three-body interaction. We show below that the typical loop momenta are of order $Q\sim\hbar/(|a|\Upsilon^{1/2})\ll\hbar/|a|$. More precision can be systematically attained by including terms in Eq. (1) with more derivatives or fields, but their effect is strongly suppressed. As we mentioned above, the interesting regime is when $g_2<0$ and $g_3>0$ [1].

Since we are interested in the condensed state of a system of bosons, it is natural to split the field ψ into a classical (c number) condensate ϕ and the fluctuations ψ: $\psi\to\psi+\phi$. (ϕ can be chosen real in this case.) In terms of the new variables, the Lagrangian density becomes

$$\begin{aligned}\mathcal{L}+\mu\mathcal{N}=&-\frac{1}{2}g_2\phi^4-\frac{1}{6}g_3\phi^6+\mu\phi^2+\phi(\psi+\psi^\dagger)\\&\times\left(\mu-g_2\phi^2-\frac{g_3}{2}\phi^4\right)+\frac{1}{2}(\psi^\dagger\psi)\\&\times\begin{pmatrix} i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m}+\chi & -g_2\phi^2-g_3\phi^4\\ -g_2\phi^2-g_3\phi^4 & -i\hbar\frac{\partial}{\partial t}+\frac{\nabla^2}{2M}+\chi\end{pmatrix}\\&\times\begin{pmatrix}\psi\\ \psi^\dagger\end{pmatrix}+(\text{interactions}),\end{aligned} \tag{4}$$

where $\chi=g_2\phi^2+g_3\phi^4$. The interaction terms include vertices with three ψ's ($\sim g_2\phi$ and $\sim g_3\phi^3$), four ψ's ($\sim g_2$ and $\sim g_3\phi^2$), five ψ's ($\sim g_3\phi$), and six ψ's ($\sim g_3$) and normal ordering is implied as well. At tree level/mean field, the particle number density is $n=\phi^2$ and the energy density and chemical potential (determined by imposing the condition $\langle\psi\rangle=0$) are given by

$$\mathcal{E}=\frac{g_2}{2}n^2+\frac{g_3}{6}n^3, \tag{5}$$

$$\mu=\frac{d\mathcal{E}}{dn}=g_2 n+\frac{g_3}{2}n^2. \tag{6}$$

In writing down the contribution quadratic in ψ's in Eq. (4), we have implicitly taken into account the condition (6). At equilibrium/zero pressure ($P=n d\mathcal{E}/dn-\mathcal{E}=0$), the values for the particle number density, energy density, and chemical potential are easily determined to be

$$n_0=\frac{1}{|a|^3}\frac{1}{\Upsilon}, \tag{7}$$

$$\mathcal{E}_0=\mu_0 n_0=-\frac{\hbar^2}{ma^2}\frac{\pi}{\Upsilon}n_0, \tag{8}$$

$$\mu_0=\left.\frac{d\mathcal{E}}{dn}\right|_{n=n_0}=\frac{\mathcal{E}_0}{n_0}=-\frac{\hbar^2}{ma^2}\frac{\pi}{\Upsilon}. \tag{9}$$

Notice that only at equilibrium, the chemical potential becomes equal to the energy per particle, see Eq. (9). The tree level/mean-field approximation is valid when the system is dilute, which holds as long as $\Upsilon\gg 1$. At and near equilibrium, the contribution of the two-body and three-body collisions are comparable in magnitude. The three-body collision term, which in dilute systems is typically negligible, is important in this case because the three-particle bound state is close to the threshold and the particles are in the Efimov regime.

From the Lagrangian (4), we can determine the normal and anomalous propagators for the Bogoliubov quasiparticle, which can be conveniently represented as a matrix:

$$D_k=\frac{1}{\omega^2-\omega_k^2}\begin{pmatrix}\omega+\frac{\hbar^2\boldsymbol{k}^2}{2m}+\chi & -\chi\\ -\chi & -\omega+\frac{\hbar^2\boldsymbol{k}^2}{2m}+\chi\end{pmatrix}, \tag{10}$$

where ω_k is the Bogoliubov dispersion relation for quasiparticle excitations,

$$\omega_k=\sqrt{\frac{\hbar^2\boldsymbol{k}^2}{2m}\left(2\chi+\frac{\hbar^2\boldsymbol{k}^2}{2m}\right)}\approx\hbar\boldsymbol{k}\sqrt{\frac{\chi}{m}}. \tag{11}$$

Since $g_2<0$, the sound velocity $s=\sqrt{\chi/m}$ for long wavelengths is imaginary for small particle number densities, namely when

$$n<n_s=\frac{|g_2|}{g_3}=\frac{2}{3}n_0 \tag{12}$$

and the homogeneous matter at these densities ($n<n_s$) is unstable towards collapse and condensation into denser droplets. For densities $n_s\leq n\leq n_0$, the pressure is negative and

such a system tends to increase its density and binding energy. If the density $n>n_0$, the internal pressure is positive and the system tends to expand towards equlibrium and lowers its energy, unless external walls exert inward pressure. Unlike the phonons in liquid ^{4}He, the curvature of the quasiparticle excitation ω_k as a function of the wave vector k is positive. As a result, these quasiparticles have a finite lifetime due to decay into two quasiparticles, corresponding to a width $\Gamma\sim k^5$ [11,16], since the velocity of a quasiparticle with finite wave vector k is larger than the sound velocity.

The quantum fluctuations can now be evaluated at the one-loop level with usual techniques. We have chosen to use dimensional regularization for the evaluation of various diverging integrals [17]. Old fashioned techniques [9–11], such as diagonalization of the quadratic part of the Lagrangian equation (4) with the a Bogoliubov transformation and subsequent evaluation of the corrections to the ground-state density and energy would lead to identical results. The particle density and the energy density at one-loop level are given by the following expressions:

$$n=\phi^2+\frac{1}{3\pi^2}\left(\frac{m\chi}{\hbar^2}\right)^{3/2}, \tag{13}$$

$$\mathcal{E}=\frac{g_2\phi^4}{2}+\frac{g_3\phi^6}{6}+\frac{8\hbar^2}{15\pi^2 m}\left(\frac{m\chi}{\hbar^2}\right)^{5/2}+\mu(n-\phi^2). \tag{14}$$

The first correction term to the energy density (and similarly to the number density as well) is formally identical to the Lee and Yang term [9–11], which was computed, however, for a Bose gas with repulsive two-body interaction. The equilibrium values for the particle number density (both condensed and noncondensed), the energy density, and energy per particle after including first-order quantum corrections become

$$n=\frac{1}{|a|^3}\frac{1}{\Upsilon}\left(1-\frac{72}{5}\alpha\right),\quad \text{where}\quad \alpha=\frac{2^{3/2}}{3\pi^{1/2}\Upsilon^{1/2}}, \tag{15}$$

$$\mathcal{E}=\mathcal{E}_0\left(1-\frac{88}{5}\alpha\right),\quad \frac{\mathcal{E}}{n}=\mu_0\left(1-\frac{16}{5}\alpha\right). \tag{16}$$

Surprisingly, Eqs. (13) and (14) determine correctly both the linear and the quadratic corrections in α for energy per particle $\mathcal{E}/n$. Therefore, the first-order quantum corrections are controlled by the same diluteness parameter $\sqrt{n|a|^3}=\Upsilon^{-1/2}\ll 1$ as in the case of a Bose gas with repulsive interaction. In spite of this formal apparent similarity, the meaning of the present result for the condensed Bose liquid when $a<0$ is qualitatively different. There is strictly no low-density expansion in the present case, since for densities $n<n_s=2n_0/3$, the system is unstable towards long-wavelength density fluctuations as the sound velocity is imaginary, see discussion of Eqs. (11) and (12).

A general argument can be given for the suppression of higher-loop corrections. Consider a generic diagram contributing to the energy density (that is, without external legs), containing I propagators, L loops, and n_i vertices with i legs. It can be estimated by

$$\begin{aligned}(\text{Energy Diagram})&\sim\left(\frac{m}{Q^2}\right)^I\left(\sqrt{\frac{\chi}{m}}QQ^3\right)^L\left(\frac{\chi}{n^{1/2}}\right)^{n_3}\left(\frac{\chi}{n}\right)^{n_4}\left(\frac{\chi}{n^{3/2}}\right)^{n_5}\left(\frac{\chi}{n^2}\right)^{n_6}\\&\sim m^{3L/2}\chi^{-I+5L/2+n_3+n_3+n_5+n_6}n^{-n_3/2-n_4-3n_5/2-2n_6},\end{aligned} \tag{17}$$

where $Q\sim\sqrt{m\chi}\sim 1/(|a|\Upsilon^{1/2})$ is the typical loop momentum [18]. When using dimensional regularization, no powers of the cutoff are present and the only momentum scale in these diagrams is set by $\sqrt{m\chi}$. Combining the above estimate with the relations

$$2I=3n_3+4n_4+5n_5+6n_6, \tag{18}$$

$$I=n_3+n_4+n_5+n_6+L-1, \tag{19}$$

we find

$$(\text{Energy Diagram})\sim m^{3L/2}\chi^{3L/2+1}n^{1-L}\sim\mathcal{E}_0\Upsilon^{-L/2}. \tag{20}$$

Consequently, up to logarithmic corrections, all higher-order loop corrections are suppressed in the dilute limit by powers of $\sqrt{n|a|^3}=\Upsilon^{-1/2}\ll 1$. As in the case of dilute Bose gases with repulsive interactions [9–11,14,15], this parameter is proportional to the ground-state fraction of non-BEC particles. In this respect, this new class of quantum liquids, in particular the boselets, is qualitatively different from liquid ^{4}He where the fraction of non-BEC particles is close to 90%.

For large $\Upsilon\gg 1$, the self-bound system is dilute. The states with positive pressure, however, can be dense. These states could still be dilute with respect to two-body collisions, however, of such a density that $|g_2|\ll g_3 n$. In this regime, the loop expansion breaks down and a further resummation is required. The energy per particle of the system is large and the system can exist only if external pressure is applied. It is very likely that the system undergoes a transi-

tion to a coherent mixture of atoms and trimers, similar to the situation put in evidence in gaseous BEC, where the coexistence of an atomic BEC with a molecular BEC was predicted [21] and experimentally observed [22]. The formation of trimers will have two effects, lowering the particle density and lowering the energy. The spatial size of a trimer when $a<0$ is smaller than $|a|$, even if the trimer binding energy is vanishing. The wave function of an Efimov state of zero energy is normalizable and the wave function is concentrated predominantly at interparticle distances $|\boldsymbol{r}_i-\boldsymbol{r}_j|<|a|$, where $i,j=1,2,3$ are particle labels in a trimer [19]. Since the average separation between atoms is presumed to be larger than the scattering length, when all atoms convert into trimers, the trimer number density decreases roughly by a factor of 3 when compared to the atomic number density [1]. The trimer phase is most likely stable, provided the trimer-trimer scattering length is positive, which is most probable [9,20]. When a trimer is formed, a significant amount of energy is released as well. These two factors, the significant drop in density and energy, should be explicitly taken into account when evaluating the energy density at higher particle densities, where triple collisions might dominate. The self-bound state we have considered is only metastable. It can decay either by a four-particle process, leading to the formation of a trimer and one energetic atom, or through a three-body process with the formation of a deeper dimer and one energetic atom (the formation of a shallow dimer is excluded since $a<0$). The rate for the first process is proportional to $\hbar n^4 a^7/m$ and likely very small. The formation of a deep dimer was considered in Ref. [23] and it was found to be of the order of $\hbar n^3 a^4/m$. This decay rate is, however, enhanced when a three-body state is close to the threshold, where the correct rate can be determined only through a nonperturbative calculation, but this is not yet attempted. This recombination rate is related to the widths of the Efimov states due to the existence of deep two-body bound states, which follow the same exponential behavior as the energies of the Efimov states [24].

Note added. Boselets made of spin-polarized tritium atoms are the likely candidates, since the regime we study here, when $g_2<0$ and $g_3>0$, could apparently be easily reached according to Ref. [25]. In this particular case, three-body recombination processes are absent.

Note added in proof. Recently Braaten and Hammar [26] performed a nonperturbative calculation of three-body recombination rates in dilute gases.

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[7] The condensation in this case has a double meaning: (1) There is a real condensation similar to the phase transition between a gas and a liquid which occurs in real space, and (2) There is Bose-Einstein condensation at temperatures close to absolute zero in momentum space.

[8] The estimate of the boselet density mentioned here (which was also mentioned in AIP, Physics News Update in July, 2002) is very rough. A better estimate could be based on Eq. (7) using $a=a_3$ and choosing $\Upsilon\approx 10$ to ensure a reasonable diluteness of the system. While the value a_3, the value of Υ is tunable. With $|a_3|=1{,}000$ Å, one obtains $n_0\approx 10^{-14}$ cm^{-3}.

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[18] Notice that the energies scale such as $\sqrt{\chi/m}Q$ is in accord with the Bogoliubov dispersion relation in the small momentum region. The cutoff may appear inside logs, so we will not be able to determine the powers of $\ln(na^3)=-\ln\Upsilon$ without a more detailed argument.

[19] The wave function of a trimer with zero energy decays at large distances ($R\gg|a|$) as $1/R^4$, where R is the hyperspherical radius. This behavior can be easily inferred from the behavior of the zero-energy three-particle propagator and by writing the

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