J. BARNARD & S. M. LUND

US PAS JANUARY 2004

I CURRENT LIMITS –

A. AXISYMMETRIC

1. SOLENOIDS

2. EINZEL LENS

B. QUADRUPOLAR

1. DERIVATION OF ENVELOPE EQUATION WITH ELLIPTICAL SYMMETRY

2. CURRENT LIMIT USING FOURIER TRANSFORM

3. ALTERNATIVE METHODS

II CENTROID

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YESTERDAY, WE DERIVED THE PARAXIAL RAY EQUATION FOR PARTICLES IN AXISYMMETRIC SYSTEMS:

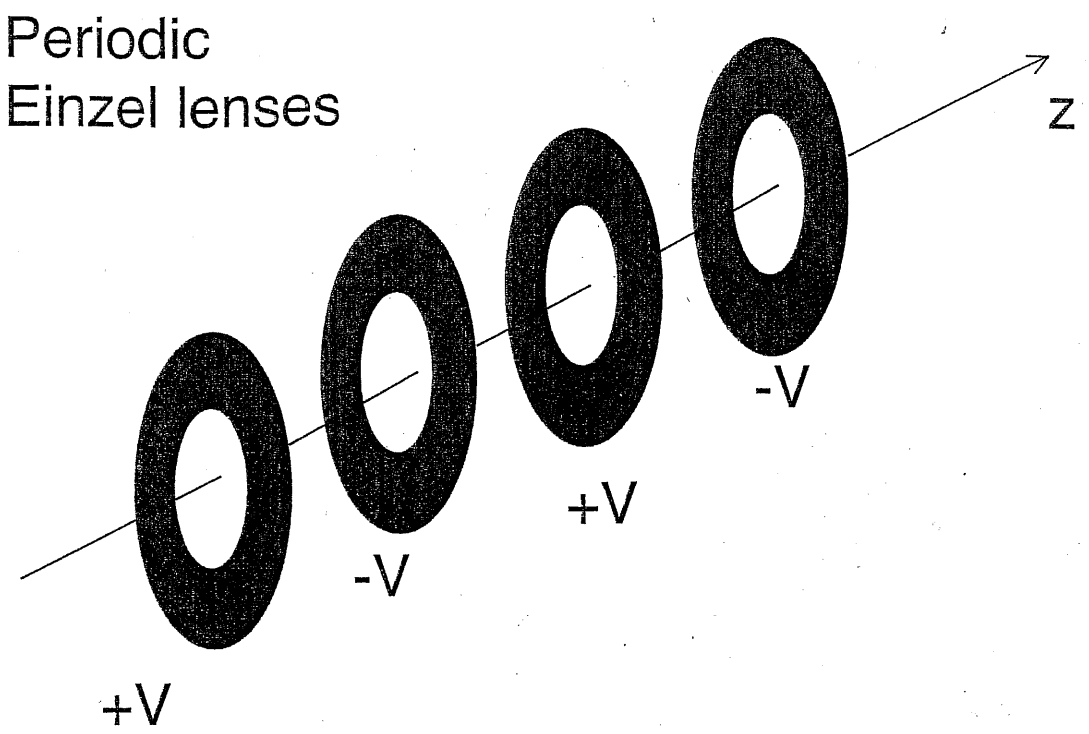
$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma}}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{\substack{V_\theta B_z \\ -\text{CENTRIFUGAL}}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\substack{\text{SELF-} \\ \text{FIELD}}} = 0$$

$$\theta' = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$ ← CONSTANCY & DEFINITION OF CANONICAL MOMENTUM

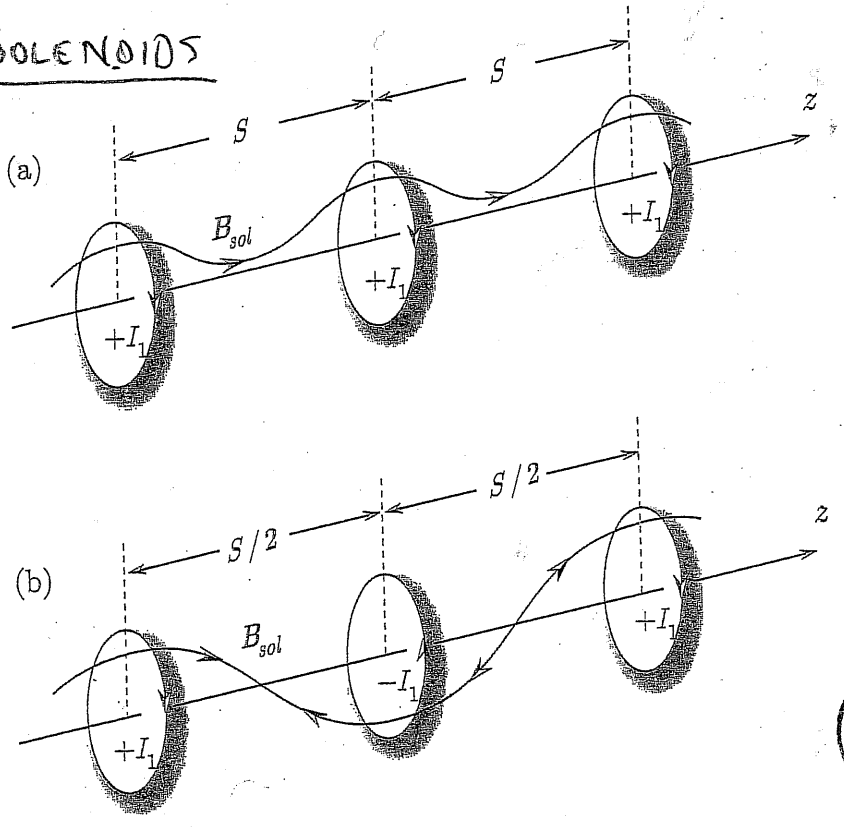
ENVELOPE EQUATION FOR AXISYMMETRIC BEAM

$$r_b'' + \frac{\gamma' r_b'}{\beta^2\gamma} + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\varepsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\varepsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$



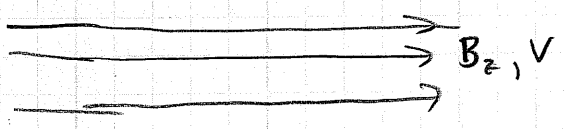
PERIODIC SOLENOIDS



(FIGURE FROM DAVIDSON & QIN, 2003) p. 55 "PHYSICS OF INTENSE CHARGED PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S. In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \cdots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \cdots$.

SOLENOIDAL FOCUSING



Let $\gamma' = \gamma'' = 0$

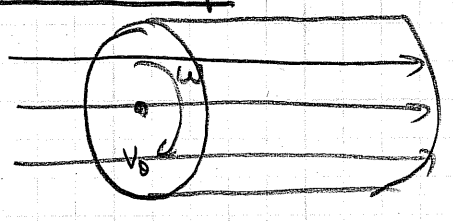
FOR MAXIMUM TRANSPORT $P_\theta = 0$ & $\varepsilon_r^2 = 0$

$$\Rightarrow \quad r_b'' + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b = \frac{Q}{r_b}$$

FOR A MATCHED BEAM:

$$Q_{max} = \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b^2$$

HEURISTICALLY:



$$V_\theta = \omega r$$

$$m\omega^2 r + Q m V_z^2 \left(\frac{r}{r_b^2}\right) = q\underbrace{\omega r}_{V_\theta} B_z$$

centrifugal force — SPACE CHARGE FORCE — MAGNETIC FORCE INWARD

$$\Rightarrow \quad \omega^2 + \frac{QV^2}{r_b^2} = \omega\omega_c$$

$\omega\omega_c - \omega^2 =$ MAXIMUM WHEN $\omega = \frac{\omega_c}{2}$

$$\Rightarrow \quad Q_{max} = \left(\frac{\omega_c^2}{4}\right)\left(\frac{r_b^2}{V^2}\right)$$

EINZEL LENS

J. BANNARD

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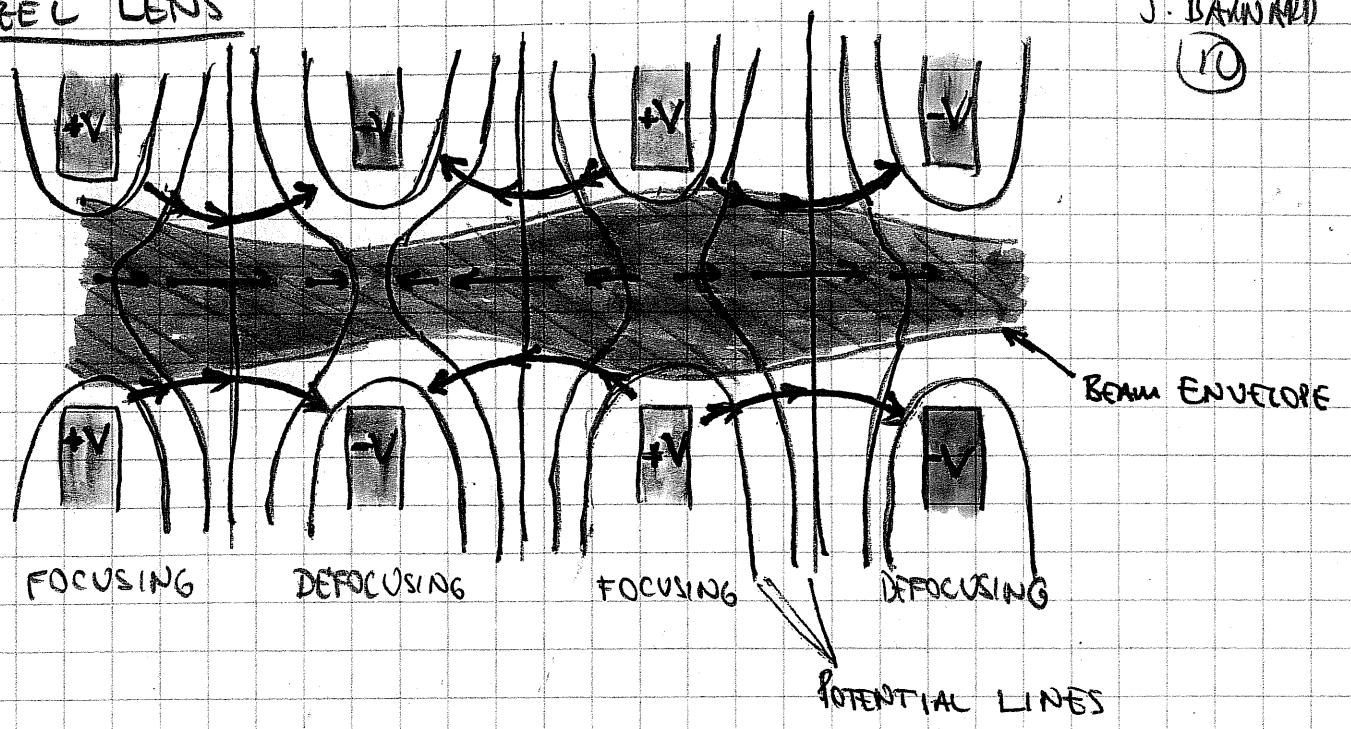

FOCUSING OCCURS AT LARGE radius THAN DEFOCUSING

$\Rightarrow$ NET INWARD FORCE

EINZEL LENS – ANALYSIS (DERIVATION FROM ED LEE)

NOW, LET $\omega_c = \langle P_\theta \rangle = \varepsilon_r^2 = 0$

$$\Rightarrow r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b - \frac{Q}{r_b} = 0$$

ALSO ASSUME $\beta \ll 1$, NON-RELATIVISTIC BEAM

$$r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

To eliminate r_b' term try substitution

$$r_b = \left(\frac{\beta_0}{\beta}\right)^{1/2} R$$

$$r_b' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R' - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-3/2} \frac{R}{\beta_0}\beta'$$

$$r_b'' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' - \left(\frac{\beta}{\beta_0}\right)^{-\frac{3}{2}} \frac{R'}{\beta_0}\beta' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-\frac{5}{2}} \frac{R}{\beta_0^2}\beta'^2 - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-\frac{3}{2}} \frac{R}{\beta_0}\beta''$$

$$\Rightarrow \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-\frac{5}{2}} \frac{\beta'^2}{\beta_0^2} R = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right)^{1/2}$$

$$\Rightarrow \boxed{R'' = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right) - \frac{3}{4}\left(\frac{\beta'}{\beta}\right)^2 R}$$

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EINZEL LENS - CONTINUED

MODEL: LET $\phi = \phi_0 \cos\left(\frac{\pi z}{L}\right)$

$$\frac{1}{2} m v^2 + q\phi = \text{constant}$$

$$\Rightarrow \quad v^2 = v_0^2 + \frac{2q\phi}{m} \cos\left(\frac{\pi z}{L}\right)$$

$$v' = \frac{-q\phi_0}{mv}\left(\frac{\pi}{L}\right) \sin\left(\frac{\pi z}{L}\right)$$

If $\left(\frac{2q\phi_0}{m}\right) << v_0^2$: $\left(\frac{\beta'}{\beta}\right)^2 \cong \left(\frac{q\phi_0}{mv_0^2}\right)^2 \left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi z}{L}\right)$

FOR EQUILIBRIUM LOOK AT D.C. COMPONENT: $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$R'' = 0 \quad \Rightarrow \quad \frac{Q}{\bar{R}} = \frac{3}{4}\overline{\left(\frac{\beta'}{\beta}\right)^2}\,\bar{R}$$

$$R = \left(\frac{\beta}{\beta_0}\right)^{1/2} r_b \quad \Rightarrow \quad \bar{R} = r_b$$

$$\overline{\left(\frac{\beta'}{\beta}\right)^2} = \frac{1}{2}\left(\frac{q\phi_0}{mv_0^2}\right)^2 \left(\frac{\pi}{L}\right)^2$$

$$\Rightarrow \quad Q_{max} = \frac{3\pi^2}{8}\left(\frac{q\phi_0}{mv_0^2}\right)^2 \left(\frac{r_b}{L}\right)^2$$

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EQUATION OF MOTION

RETURN TO X, Y COORDINATES

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)x' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial x} \pm \begin{cases} \frac{qB'}{\gamma m v_z}x & \text{for magnetic quads} \\ \frac{qE'}{\gamma m v_z^2}x & \text{for electric quads} \end{cases}$$

Let $\frac{\gamma m v_z}{q} = \frac{p}{q} \equiv [B\rho] \equiv$ RIGIDITY

$$y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)y' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial y} \mp \begin{cases} \frac{B'}{[B\rho]}y & \text{magnetic} \\ \frac{qE'}{\gamma m v_z^2}y & \text{electric} \end{cases}$$

ENVELOPE EQUATION

$$r_x^2 = 4\langle x^2\rangle; \quad r_y^2 = 4\langle y^2\rangle$$

$$r_x' = \frac{4\langle x x'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}; \qquad \varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right)$$

$$r_y'' = \frac{4\langle y y''\rangle}{r_y} + \frac{\varepsilon_y^2}{r_y^3} \qquad \varepsilon_y^2 = 16\left(\langle y^2\rangle\langle y'^2\rangle - \langle y y'\rangle^2\right)$$

for magnetic focusing:

$$r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)r_x' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle x\frac{\partial\varphi}{\partial x}\rangle}{r_x} \pm \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)r_y' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle y\frac{\partial\varphi}{\partial y}\rangle}{r_y} \pm \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_y^2}{r_y^3} = 0$$

(for electric focusing $\frac{B'}{[B\rho]} \to \frac{qE'}{\gamma m v_z^2}$)

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SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY

NOW DEFOCUSING IN ONE DIRECTION AND FOCUSING IN THE OTHER $\Rightarrow$ RADIAL SYMMETRY SHOULD BE REPLACED BY ELLIPTICAL SYMMETRY:

$$\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

CAN BE SHOWN THAT (Sacherer, 1971)

$$\left\langle x \frac{\partial \varphi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$$

&

$$\left\langle y \frac{\partial \varphi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$$

OUTLINE OF PROOF: (from R. Ryne)

Let $\chi = \dfrac{x^2}{r_x^2+s} + \dfrac{y^2}{r_y^2+s}$

DEFINE $\eta(\chi)$ such that $\rho(x,y) = \left.\dfrac{d\eta(\chi)}{d\chi}\right|_{s=0} = \hat{\rho}(\chi)\big|_{s=0}$

So $\rho = \hat{\rho}\left(\dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2}\right) = \hat{\rho}(\chi)\big|_{s=0}$

DEFINE $\Psi(x,y) = \dfrac{-r_x r_y}{4\varepsilon_0} \displaystyle\int_0^\infty \frac{\eta(\chi)\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$

It follows that $\dfrac{\partial^2\Psi}{\partial x^2} + \dfrac{\partial^2\Psi}{\partial y^2} = -\dfrac{\rho}{\varepsilon_0}$ AND SO IS A SOLUTION OF POISSON'S EQUATION (since $\Psi \to 0$ as $x, y \to \infty$)

WHAT IS $\left\langle x \dfrac{\partial\varphi}{\partial x}\right\rangle$?

$$\left\langle x \frac{\partial\Psi}{\partial x}\right\rangle = \frac{-r_x r_y}{4\lambda\varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, x\, \rho(x,y) \int_0^\infty \frac{\eta' \frac{\partial\chi}{\partial x}\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$$

where $\lambda = \displaystyle\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \rho(x,y)$

So $$\left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-2 r_x r_y}{4 \lambda \varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; x^2 \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) \int_0^{\infty} \frac{\hat{\rho}\left(\frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}\right) ds}{(r_x^2+s)^{3/2}(r_y^2+s)^{1/2}}$$

Let $$r\cos\theta = \frac{x}{\sqrt{r_x^2+s}} \qquad r\sin\theta = \frac{y}{\sqrt{r_y^2+s}}$$

$$\det J = \sqrt{r_x^2+s}\,\sqrt{r_y^2+s}\; r$$ where J is the Jacobian $dx\,dy = \det J \cdot dr\,d\theta$

$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x r_y}{\lambda 2 \varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_0^{\infty} ds \; r^3 \hat{\rho}(r^2)\, \rho\left(\frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2 \sin^2\theta\right)$$

Let $$r'^2 = \frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2 \sin^2\theta$$
$$= r^2\left[1 + s\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)\right]$$

with r fixed $$2r'dr' = r^2\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right) ds$$

$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x r_y}{2\lambda \varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_r^{\infty} dr' \; \frac{2r'dr'\, r^3 \rho(r^2)}{r^2\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)} \hat{\rho}(r'^2) \cos^2\theta$$

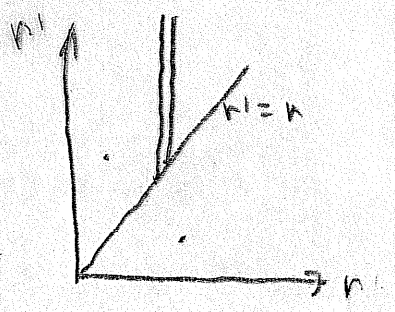
$$\int_0^{2\pi} \frac{\cos^2\theta \, d\theta}{\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}} = \frac{2\pi r_x^2 r_y}{r_x + r_y}$$

$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x^3 r_y^2}{\lambda 2\pi \varepsilon_0 (r_x + r_y)} \int_0^{\infty} dr\, 2\pi r\, \hat{\rho}(r^2) \int_r^{\infty} dr'\, 2\pi r' \hat{\rho}(r'^2)$$

Recall $$\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, \rho(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

Let $$\frac{x}{r_x} = r\cos\theta \qquad \frac{y}{r_y} = r\sin\theta \qquad \det J = r_x r_y r$$
$$\Rightarrow \lambda = \int_0^{\infty} dr \int_0^{2\pi} d\theta\, \rho(r^2)\, r_x r_y r = 2\pi r_x r_y \int_0^{\infty} dr\, r\, \rho(r^2)$$

Now $\int_0^\infty dr \int_r^\infty dr' \, \hat{\rho}(r^1)\hat{\rho}(r'^2) = \frac{1}{2}\underbrace{\int_0^\infty dr \, r \, \hat{\rho}(r^1)}_{\lambda/2\pi r_x r_y} \underbrace{\int_0^\infty dr' \, r' \hat{\rho}(r'^2)}_{\lambda/2\pi r_x r_y}$

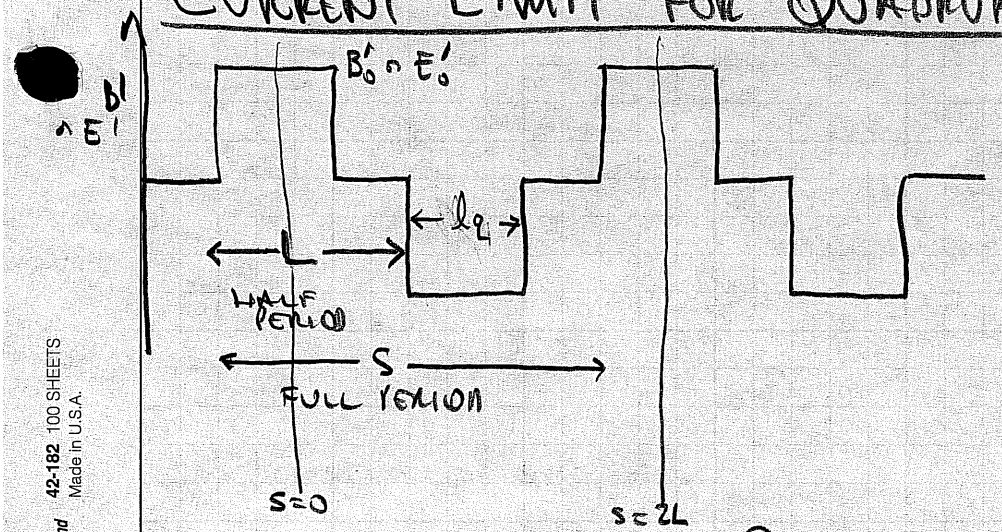


$\hat{\rho}(r^1)\hat{\rho}(r^2)$ is the same on switching r & r'.

So $$\left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x^3 r_y}{\lambda 2\pi\varepsilon_0 (r_x + r_y)} \left[\frac{1}{2} \frac{\lambda}{2\pi r_x r_y} \frac{\lambda}{2\pi r_x r_y} \right]$$

$$= \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$$

CURRENT LIMIT FOR QUADRUPOLES



$$k = \begin{cases} \dfrac{B_0'}{[B\rho]} & \text{MAGNETIC} \\ \dfrac{qE_0'}{\gamma m v_z^2} & \text{ELECTRIC} \end{cases}$$

$$r_x'' + k f(s) r_x - \frac{2Q}{r_x + r_y} = 0$$

(NOTE WE HAVE SET $\epsilon = 0$).

$$r_y'' - k f(s) r_y - \frac{2Q}{r_x + r_y} = 0$$

$$f(s) = \begin{cases} 1 & 0 < s < nL/2 \\ -1 & L - \frac{nL}{2} < s < L + nL/2 \\ 1 & 2L - nL/2 < s < 2L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2L}\int_0^{2L} f(s)\cos\left(\frac{\pi s}{L}\right) ds = \left(\frac{2}{\pi}\right)\sin\left(\frac{\pi n}{2}\right)$$

= fourier amplitude at fundamental lattice frequency

Let $r_x = r_b\left(1 + \delta\cos\left(\frac{\pi s}{L}\right)\right)$

$r_y = r_b\left(1 - \delta\cos\left(\frac{\pi s}{L}\right)\right)$

Let $k f(s) \rightarrow k\left(\frac{2}{\pi}\right)\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi s}{L}\right)$

COLLECTING "FAST" TERMS AND "SLOW" TERMS

$$\left[-\frac{\pi^2}{L^2} r_b \delta + k r_b\left(\frac{2\sin\left(\frac{n\pi}{2}\right)}{\pi}\right)\right]\cos\left(\frac{\pi s}{L}\right) = 0 \qquad \text{(fast)}$$

$$\delta k r_b\left(\frac{2\sin\left(\frac{n\pi}{2}\right)}{\pi}\right) = \frac{Q}{r_b} \qquad \text{(slow)}$$

Fast $\Rightarrow$ $\delta = \dfrac{2kL^2}{\pi^3}\sin\left(\dfrac{n\pi}{2}\right)$ & $Q_{max} \cong \dfrac{n^2 k^2 L^2}{\pi^2}\left(\dfrac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right)^2 r_b^2$

CONTINUOUS FOCUSING

ASSUME AN UNSPECIFIED radial force

$$x'' = \frac{-\sigma_0^2}{4L^2}x + \frac{Q_x}{r_b^2}; \quad y'' = \frac{-\sigma_0^2}{4L^2}y + \frac{Q_y}{r_b^2};$$ HERE $L \equiv$ HALF-LATTICE PERIOD (IN ANALOGY TO QUADRUPOLE FOCUSING)

$$\Rightarrow \quad r_x'' + \frac{\sigma_0^2}{4L^2}r_x - \frac{2Q}{r_x+r_y} - \frac{\epsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{\sigma_0^2}{4L^2}r_y - \frac{2Q}{r_x+r_y} - \frac{\epsilon_y^2}{r_y^3} = 0$$

[CONTINUOUS focusing could represent the radial focusing of a uniform static charge density $\rho = \frac{-\sigma_0^2}{2L^2}\left(\frac{\gamma m v_z^2}{q}\right)\varepsilon_0$, for example or in an approximate manner the average effects of periodic focusing.]

NOTE THAT FOR CONTINUOUS FOCUSING THE DEFINESED TUNE

$$\frac{\sigma^2}{4L^2} = \frac{\sigma_0^2}{4L^2} - \frac{Q}{r_b^2}$$

NOTE THAT FOR A MATCHED BEAM $r_x = r_y = r_b$.

NOTE ALSO THAT FOR A MATCHED BEAM:

$$\frac{\sigma^2}{4L^2} = \frac{\epsilon^2}{r_b^4} \quad \text{or} \quad \frac{\sigma}{2L} = \frac{\epsilon}{r_b^2}$$

$$\Rightarrow \quad \frac{\sigma^2}{\sigma_0^2} = \frac{1}{1 + \frac{Q r_b^2}{\epsilon^2}}$$

ANOTHER WAY OF LOOKING AT CURRENT LIMIT:

$$\frac{Q}{r_b} = \frac{\sigma_0^2}{4L^2} r_b$$

where σ_0 is single particle phase advance

SPACE CHARGE

CONTINUOUS FOCUSING

$$Q_{max} = \frac{\sigma_0^2}{4L^2} r_b^2$$

THIN LENS RESULT:

$$\frac{\sigma_0}{2L} = \left(\frac{\eta L}{2}\right) k \quad \Rightarrow \quad Q_{max} = \frac{\eta^2 L^2}{4} k^2 r_b^2$$

$$\sigma_0 \cong \eta L^2 k$$

ELIMINATING L

$$\Rightarrow \quad Q_{max} = \frac{\eta \sigma_0 k}{4} r_b^2$$

EVEN MORE ACCURATE: Lee, Fessenden, & Laslett
IEEE Trans. on Nuclear Science
NS-32, 2489 (1985)

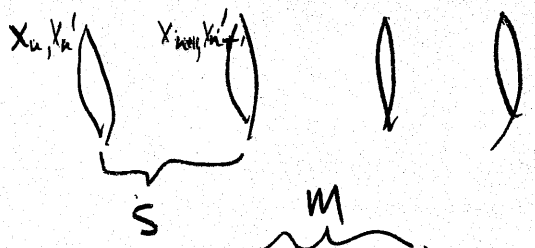
[BASED ON TRANSFER MATRIX RESULTS]

$$Q_{max} \cong \frac{(1-\cos\sigma_0)}{2L^2} r_b^2 + \cdots$$

$$\cos\sigma_0 \cong 1 - \frac{\eta^2 k^2 L^4}{2}\left[1 - \frac{2\eta}{3}\right]$$

$$\Rightarrow \boxed{Q_{max} = \frac{\eta\sqrt{1-2\eta/3}\,\sqrt{2(1-\cos\sigma_0)}\,k}{4} r_b^2}$$

Periodic Focusing (CALCULATING σ_0)



$$\begin{pmatrix} x_{n+1} \\ x_{n+1}' \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{M} \begin{pmatrix} x_n \\ x_n' \end{pmatrix}$$

USE TRANSFER MATRIX TO EXPRESS X AT POSITION n, n+1, n+2 (EACH POSITION SEPARATED BY S)

$$x_{n+1} = a x_n + b x_n' \qquad x_n' = \frac{1}{b}(x_{n+1} - a x_n)$$
$$x_{n+1}' = c x_n + d x_n'$$

$$\begin{aligned} x_{n+2} &= a x_{n+1} + b x_{n+1}' \\ &= a x_{n+1} + bc x_n + bd x_n' \\ &= a x_{n+1} + bc x_n + bd\left(\frac{1}{b}\right)(x_{n+1} - a x_n) \\ &= (a+d) x_{n+1} + (bc - ad) x_n \end{aligned}$$

$$\Rightarrow \boxed{x_{n+2} - (\mathrm{Tr}\, M)\, x_{n+1} + \det M\, x_n = 0}$$

Let $x_n = x_0 e^{in\sigma_0} \Rightarrow x_{n+1} = x_0 e^{i(n+1)\sigma_0} \qquad x_{n+2} = x_0 e^{i(n+2)\sigma_0}$

$$\Rightarrow x_0 e^{in\sigma_0 + 2i\sigma_0} - (\mathrm{Tr}\, M)\, x_0 e^{in\sigma_0 + i\sigma_0} + \det M\, x_0 e^{in\sigma_0} = 0$$

$$\Rightarrow e^{2i\sigma_0} - \mathrm{Tr}\, M\, e^{i\sigma_0} + 1 = 0 \qquad (\text{assuming } \det M = 1)$$

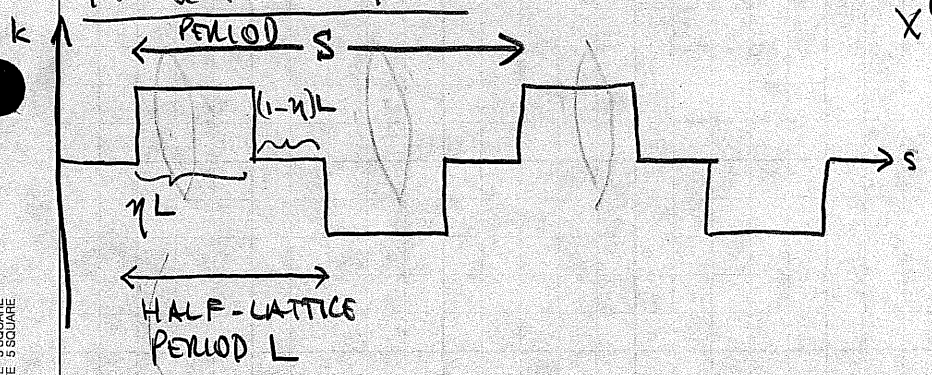
$$\Rightarrow \boxed{\mathrm{Tr}\, M = \frac{e^{2i\sigma_0} + 1}{e^{i\sigma_0}} = e^{i\sigma_0} + e^{-i\sigma_0} = 2\cos\sigma_0}$$

Example:

$$x'' = -kx \rightarrow \begin{pmatrix} x_{n+1} \\ x_{n+1}' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{k}S & \frac{1}{\sqrt{k}}\sin\sqrt{k}S \\ -\sqrt{k}\sin\sqrt{k}S & \cos\sqrt{k}S \end{pmatrix} \begin{pmatrix} x_n \\ x_n' \end{pmatrix}$$

$$\cos\sigma_0 = \cos\sqrt{k}S \qquad \Rightarrow \sigma_0 = \sqrt{k}S$$

For a FODO system:



$$x'' = k(s)\,x$$

$$k = \begin{cases} B'/[B\rho] & \text{(magnetic)} \\ qE'/\gamma m v_z^2 & \text{(electric)} \end{cases}$$

$$\underbrace{\begin{pmatrix} \cos\sqrt{k}\,\eta L & \frac{1}{\sqrt{k}}\sin\sqrt{k}\,\eta L \\ -\sqrt{k}\sin\sqrt{k}\,\eta L & \cos\sqrt{k}\,\eta L \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} 1 & (1-\eta)L \\ 0 & 1 \end{pmatrix}}_{O} \underbrace{\begin{pmatrix} \cosh\sqrt{k}\,\eta L & \frac{1}{\sqrt{k}}\sinh\sqrt{k}\,\eta L \\ \sqrt{k}\sinh\sqrt{k}\,\eta L & \cosh\sqrt{k}\,\eta L \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & (1-\eta)L \\ 0 & 1 \end{pmatrix}}_{O}$$

Let $\theta = \sqrt{k}\,\eta L$

$$\cos\sigma_0 = \cos\theta\cosh\theta + \frac{(1-\eta)}{\eta}\theta(\cos\theta\sinh\theta - \sin\theta\cosh\theta) - \frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\theta^2\sin\theta\sinh\theta$$

for $\theta \ll 1$

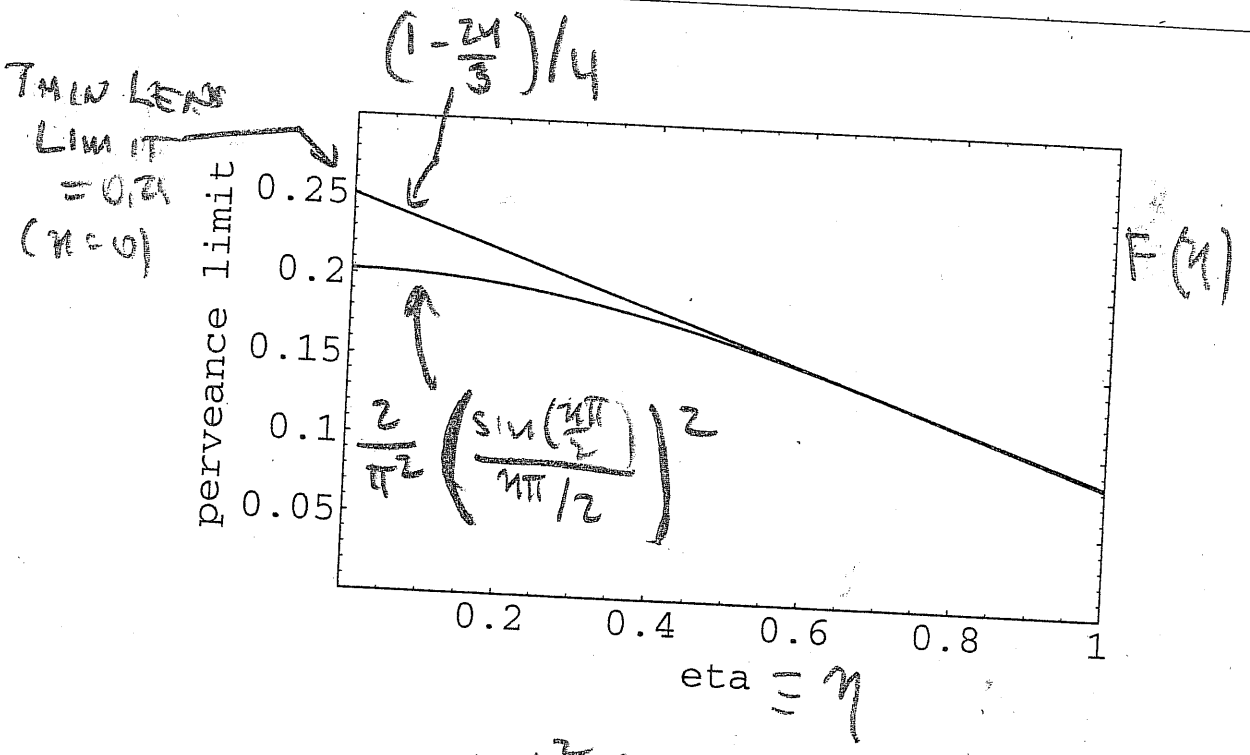
$$\cos\sigma_0 \cong 1 - \frac{1}{2}\left(1 - \frac{2}{3}\eta\right)\frac{\theta^4}{\eta^2}$$

$\rightarrow$ If $\sigma_0 \ll 1$ & $\eta \ll 1$

$$\sigma_0 \simeq \frac{\theta^2}{\eta} = k\eta L^2$$

$$k_{\beta 0} = \frac{\sigma_0}{2L} = \left(\frac{\eta L}{2}\right)k$$

$$x'' \cong -k_{\beta 0}^2\,x \cong -\left(\frac{\eta L}{2}\right)^2 k^2\,x$$



$$Q_{max} = F(\eta)\,\eta^2 k^2 L^2 r_b^2$$

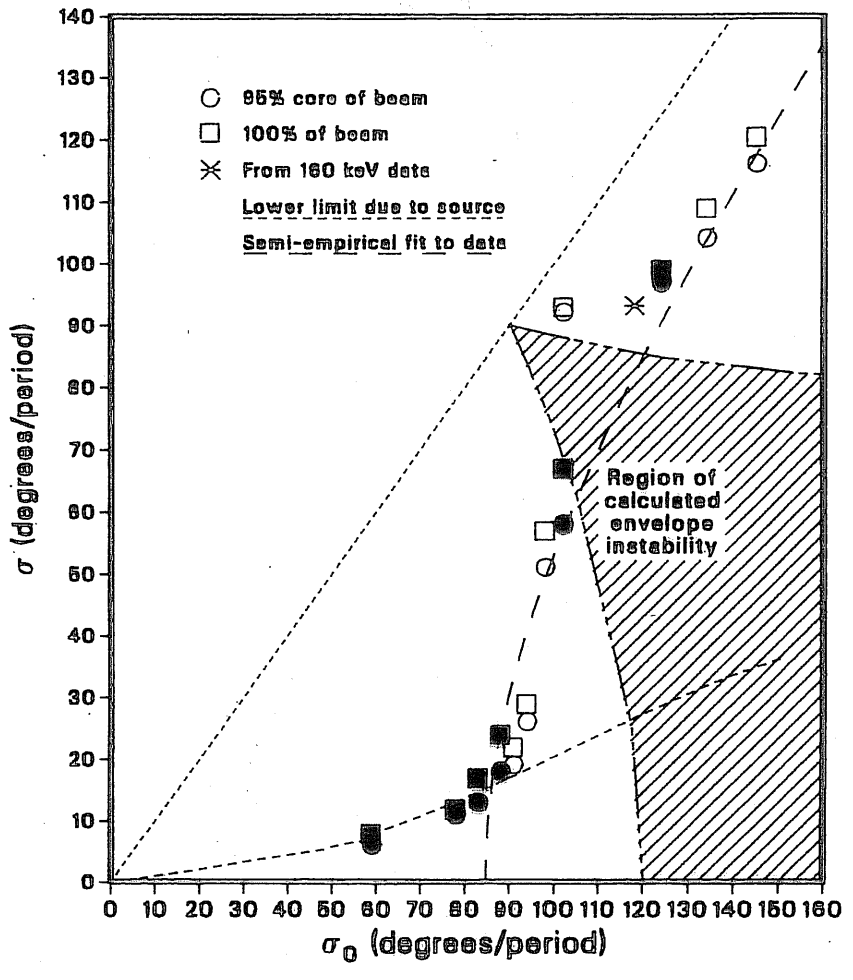
$$F(\eta) = \begin{cases} \frac{1}{4} & \text{THIN LENS, WEAK } Q \\ \frac{2}{\pi^2}\left(\frac{\sin\left[\frac{\eta\pi}{2}\right]}{\frac{\eta\pi}{2}}\right)^2 & \text{FOURIER COMPONENT AVIN (CAROKI} \\ \left(1-\frac{2\eta}{3}\right)/4 & \text{LEE, FESSENDEN, LAILETT} \end{cases}$$

Envelope instabilities set upper limit on "single particle" phase advance σ_0

Experimental data (Tiefenback, 1986) from LBL's Single Beam Transport Experiment (SBTE)

$\sigma_0 < 85^o$

Experimental limits on beam stability in terms of σ and σ_0



J. BARNARD

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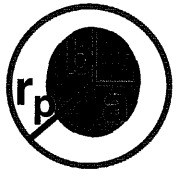
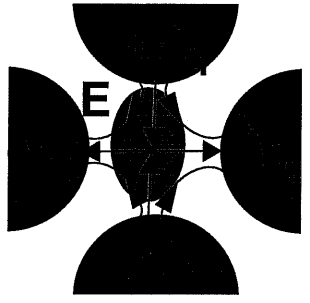
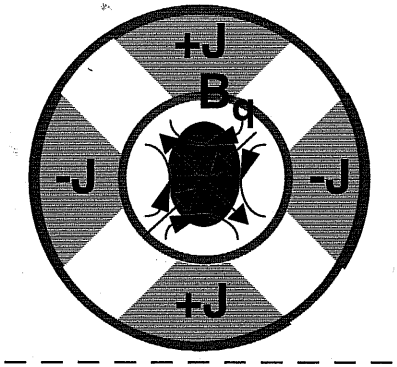
Focusing systems scale differently w/ ion energy qV, charge/mass ratio q/m, and lattice period 2L



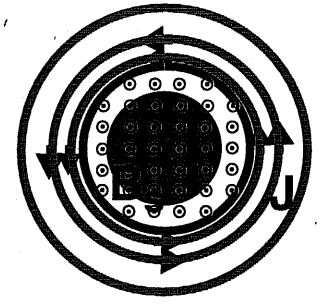
Electric quads



Magnetic quads



Solenoids



Envelope equation:

$$r_b'' = \frac{\varepsilon^2}{r_b^3} + \frac{2K}{r_x + r_y} \pm \frac{V_q}{V}\frac{r_b}{r_p^2} \qquad \frac{2K}{a+b} \pm \left(\frac{qB_q^2}{2mV}\right)^{1/2}\frac{r_b}{r_p} \qquad \frac{K}{r_b} + \frac{\omega^2 r_b}{v_z^2} - \frac{\omega\omega_c r_b}{v_z^2}$$

smooth limit, averaging over L

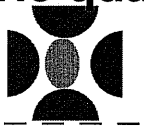
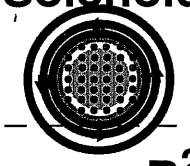
setting $\omega=\omega_c/2$

$$r_b'' = \frac{\varepsilon^2}{r_b^3} + \frac{K}{r_b} - k^2 r_b$$

$$k^2 = \left(\frac{\sigma_0}{2L}\right)^2 \cong \frac{1}{4r_p^2}\left(\frac{\eta L}{r_p}\right)^2\frac{V_q^2}{V^2} \qquad \frac{1}{8}\left(\frac{\eta L}{r_p}\right)^2\left(\frac{qB_q^2}{mV}\right) \qquad \frac{\eta}{8}\left(\frac{qB_s^2}{mV}\right)$$

(where K= perveance=$\lambda/4\pi\varepsilon_0 V$, ω=rotation freq.,ω_c = cyclotron freq., σ_0 = undepressed phase advance., ε = emittance, and $mv_z^2/2 = qV$)

Scaling of line charge density λ_b with ion energy qV, charge/mass ratio q/m, and lattice period 2L

Electric quads	Magnetic quads	Solenoids
		

$$k^2 = \left(\frac{\sigma_0}{2L}\right)^2 \cong \frac{1}{4r_p^2}\left(\frac{\eta L}{r_p}\right)^2 \frac{V_q^2}{V^2} \qquad \frac{1}{8}\left(\frac{\eta L}{r_p}\right)^2\left(\frac{qB_q^2}{mV}\right) \qquad \frac{\eta}{8}\left(\frac{qB_s^2}{mV}\right)$$

Envelope/lattice instability limit: $\sigma_0 \lesssim \pi/2$

Maximum line charge density per beam λ_b (found from $Q/r_b = k^2 r_b$)

$$0.9\ \frac{\mu C}{m}\left(\frac{V_q}{80\ kV}\right)\left(\frac{\sigma_0}{1.4}\right)\left(\frac{r_b/r_p}{0.7}\right)^2\left(\frac{\eta}{0.7}\right)$$

$$\lambda_b = 1\ \frac{\mu C}{m}\left(\frac{\eta}{0.7}\right)\left(\frac{\sigma_0}{1.4}\right)\left(\frac{r_b/r_p}{0.7}\right)^2\left(\frac{B_q}{2T}\right)\left(\frac{q/m}{1/200}\right)^{1/2}\left(\frac{V}{2MeV}\right)^{1/2}\left(\frac{r_p}{6cm}\right)$$

$$.03\ \frac{\mu C}{m}\left(\frac{\eta}{0.7}\right)\left(\frac{r_b/r_p}{0.7}\right)^2\left(\frac{B_s}{2T}\right)^2\left(\frac{q/m}{1/200}\right)\left(\frac{r_p}{6cm}\right)^2$$

for (magnetic quads):

$$I_b = \lambda_b v = 1.4\ A\left(\frac{\eta}{0.7}\right)\left(\frac{\sigma_0}{1.4}\right)\left(\frac{r_b/r_p}{0.7}\right)^2\left(\frac{B_q}{2T}\right)\left(\frac{q/m}{1/200}\right)\left(\frac{V}{2MeV}\right)\left(\frac{r_p}{6cm}\right)$$

FIRST ORDER MOMENTS DESCRIBE "CENTROID" EVOLUTION

$$x'' = -\frac{\sigma_0^2}{4L^2}x + \frac{Q(x-\langle x\rangle)}{r_b^2}$$

$$\frac{d}{ds}\langle x\rangle = \langle x'\rangle$$

$$\frac{d}{ds}\langle x'\rangle = \langle x''\rangle = -\frac{\sigma_0^2}{4L^2}\langle x\rangle + \frac{Q}{r_b^2}\cancelto{0}{(\langle x\rangle - \langle x\rangle)}$$

$$\Rightarrow \frac{d^2\langle x\rangle}{ds^2} = -\frac{\sigma_0^2}{4L^2}\langle x\rangle$$

$\Rightarrow$ MOTION OF CENTROID IS UNAFFECTED BY SPACE-CHARGE <u>EXCEPT</u> THROUGH IMAGE CHARGES

IMAGE FORCES (SUMMARY ONLY)

cf. REISER SEC 4.4.4

To linear order:

$$\frac{d^2\langle x\rangle}{ds^2} \cong -\left(\frac{\sigma_0^2}{4L^2} - \frac{Q}{r_p^2}\right)\langle x\rangle$$

IMAGES ATTRACT BEAM TO PIPE REDUCING FOCUSING FIELD

ENVELOPES ARE ALSO EFFECTED BY IMAGES (cf eg. Lee, CLOSE, & SMITH, PROC. 1987 IEEE PAC, p. 1126)

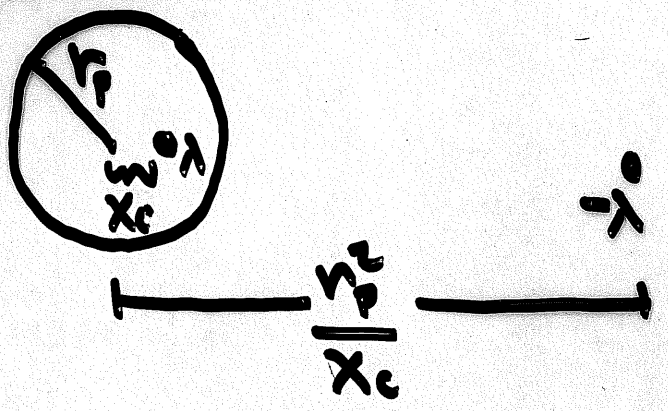
To lowest order:

$$\frac{d^2 r_x}{ds^2} = \cdots + \frac{2Q}{r_x + r_y} + \frac{fQ}{r_p^2} r_x$$

$$\frac{d^2 r_y}{ds^2} = \cdots + \frac{2Q}{r_x + r_y} + \frac{fQ}{r_p^2} r_y$$

$$f = \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{\langle x\rangle^2}{r_p^2}\left[1 + \frac{3}{2}\frac{(r_x^2 - r_y^2)}{r_p^2} + \frac{3}{8}\frac{(r_x^2 - r_y^2)^2}{r_p^4}\right]$$

Image Force



$$E_x = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r_p^2/x_c - x_c} \simeq \frac{\lambda}{2\pi\epsilon_0}\frac{x_c}{r_p^2}$$ [redacted]

Image currents also can give rise to a

$$|B_y| = \beta E_x$$

$$F_x = (1-\beta^2)\lambda E_x$$

so that $x_c'' \cong -\left(\frac{a_0^2}{4l^2} - \frac{Q}{r_p^2}\right)x_c$